Caltech Harvey Mudd Mathematics Competition

Individual Round **February 20, 2010**

1. Compute the degree of the least common multiple of the polynomials $x-1,\ x^2-1,\ x^3-1,\dots,\ x^{10}-1$.

2. A line in the xy plane is called *wholesome* if its equation is $y = mx + b$ where m is rational and b is an integer. Given a point with integer coordinates (x, y) on a wholesome line l, let r be the remainder when x is divided by 7, and let s be the remainder when y is divided by 7. The pair (r, s) is called an *ingredient* of the line l. The (unordered) set of all possible ingredients of a wholesome line l is called the *recipe* of l. Compute the number of possible recipes of wholesome lines.

3. Let $\tau(n)$ be the number of distinct positive divisors of n. Compute $\sum_{d|15015} \tau(d)$, that is, the sum of $\tau(d)$ for all d such that d divides 15015.

4. Suppose $2202010_b - 2202010_3 = 71813265_{10}$. Compute b. (n_b denotes the number n written in base b.)

5. Let $x = (3 - \sqrt{5})/2$. Compute the exact value of $x^8 + 1/x^8$.

6. Compute the largest integer that has the same number of digits when written in base 5 and when written in base 7. Express your answer in base 10.

7. Three circles with integer radii a, b, c are mutually externally tangent, with $a \le b \le c$ and $a < 10$. The centers of the three circles form a right triangle. Compute the number of possible ordered triples (a, b, c).

8. Six friends are playing informal games of soccer. For each game, they split themselves up into two teams of three. They want to arrange the teams so that, at the end of the day, each pair of players has played at least one game on the same team. Compute the smallest number of games they need to play in order to achieve this.

9. Let A and B be points in the plane such that $AB = 30$. A circle with integer radius passes through A and B. A point C is constructed on the circle such that $\overline{AC}$ is a diameter of the circle. Compute all possible radii of the circle such that BC is a positive integer.

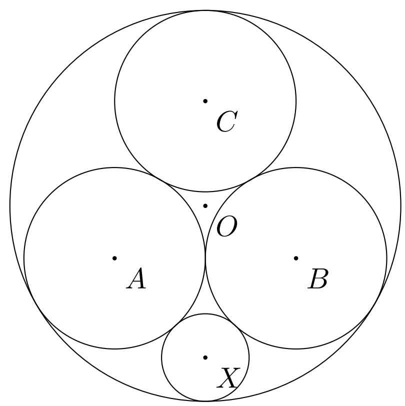
10. Each square of a 3×3 grid can be colored black or white. Two colorings are the same if you can rotate or reflect one to get the other. Compute the total number of unique colorings.

11. Compute all positive integers n such that the sum of all positive integers that are less than n and relatively prime to n is equal to $2n$.

12. The distance between a point and a line is defined to be the smallest possible distance between the point and any point on the line. Triangle ABC has $AB = 10$, $BC = 21$, and $CA = 17$. Let P be a point inside the triangle. Let x be the distance between P and $\overleftrightarrow{BC}$, let y be the distance between P and $\overleftrightarrow{CA}$, and let z be the distance between P and $\overleftrightarrow{AB}$. Compute the largest possible value of the product xyz.

13. Alice, Bob, David, and Eve are sitting in a row on a couch and are passing back and forth a bag of chips. Whenever Bob gets the bag of chips, he passes the bag back to the person who gave it to him with probability $\frac{1}{3}$, and he passes it on in the same direction with probability $\frac{2}{3}$. Whenever David gets the bag of chips, he passes the bag back to the person who gave it to him with probability $\frac{1}{4}$, and he passes it on with probability $\frac{3}{4}$. Currently, Alice has the bag of chips, and she is about to pass it to Bob when Cathy sits between Bob and David. Whenever Cathy gets the bag of chips, she passes the bag back to the person who gave it to her with probability p, and passes it on with probability $1 - p$. Alice realizes that because Cathy joined them on the couch, the probability that Alice gets the bag of chips back before Eve gets it has doubled. Compute p.

14. Circle O is in the plane. Circles A, B, and C are congruent, and are each internally tangent to circle O and externally tangent to each other. Circle X is internally tangent to circle O and externally tangent to circles A and B. Circle X has radius 1. Compute the radius of circle O.



15. Compute the number of primes p less than 100 such that p divides $n^2 + n + 1$ for some integer n.