Mixer Round

Caltech Harvey Mudd Mathematics Competition

February 20, 2010

1. Compute x such that $2009^{2010} \equiv x \pmod{2011}$ and $0 \leq x < 2011$.

2. Compute the number of "words" that can be formed by rearranging the letters of the word "syzygy" so that the y's are evenly spaced. (The y's are *evenly spaced* if the number of letters (possibly zero) between the first y and the second y is the same as the number of letters between the second y and the third y.)

3. Let A and B be subsets of the integers, and let $A+B$ be the set containing all sums of the form $a+b$, where a is an element of A, and b is an element of B. For example, if $A = \{0, 4, 5\}$ and $B = \{-3, -1, 2, 6\}$, then $A + B = \{-3, -1, 1, 2, 3, 4, 6, 7, 10, 11\}$. If A has 1955 elements and B has 1891 elements, compute the smallest possible number of elements in $A + B$.

4. Compute the sum of all integers of the form p^n where p is a prime, $n \geq 3$, and $p^n \leq 1000$.

5. In a season of interhouse athletics at Caltech, each of the eight houses plays each other house in a particular sport. Suppose one of the houses has a $1/3$ chance of beating each other house. If the results of the games are independent, compute the probability that they win at least three games in a row.

6. A positive integer n is *special* if there are exactly 2010 positive integers smaller than n and relatively prime to n. Compute the sum of all special numbers.

7. Eight friends are playing informal games of ultimate frisbee. For each game, they split themselves up into two teams of four. They want to arrange the teams so that, at the end of the day, each pair of players has played at least one game on the same team. Determine the smallest number of games they need to play in order to achieve this.

8. Compute the number of ways to choose five nonnegative integers a, b, c, d, and e, such that $a + b + c + d + e = 20$.

9. Is 23 a square mod 41? Is 15 a square mod 41?

10. Let $\phi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute $\sum_{d|15015} \phi(d)$.

11. Compute the largest possible volume of an regular tetrahedron contained in a cube with volume 1.

12. Compute the number of ways to cover a 4×4 grid with dominoes.

13. A collection of points is called *mutually equidistant* if the distance between any two of them is the same. For example, three mutually equidistant points form an equilateral triangle in the plane, and four mutually equidistant points form a regular tetrahedron in three-dimensional space. Let A, B, C, D, and E be five mutually equidistant points in four-dimensional space. Let P be a point such that $AP = BP = CP = DP = EP = 1$. Compute the side length AB.

14. Ten turtles live in a pond shaped like a 10-gon. Because it's a sunny day, all the turtles are sitting in the sun, one at each vertex of the pond. David decides he wants to scare all the turtles back into the pond. When he startles a turtle, it dives into the pond. Moreover, any turtles on the two neighbouring vertices also dive into the pond. However, if the vertex opposite the startled turtle is empty, then a turtle crawls out of the pond and sits at that vertex. Compute the minimum number of times David needs to startle a turtle so that, by the end, all but one of the turtles are in the pond.

15. The game *hexapawn* is played on a 3×3 chessboard. Each player starts with three pawns on the row nearest him or her. The players take turns moving their pawns. Like in chess, on a player's turn he or she can either

    - move a pawn forward one space if that square is empty, or
    - capture an opponent's pawn by moving his or her own pawn diagonally forward one space into the opponent's pawn's square.

    A player wins when either

    - he or she moves a pawn into the last row, or
    - his or her opponent has no legal moves.

    Eve and Fred are going to play hexapawn. However, they're not very good at it. Each turn, they will pick a legal move at random with equal probability, with one exception: If some move will immediately win the game (by either of the two winning conditions), then he or she will make that move, even if other moves are available. If Eve moves first, compute the probability that she will win.