Caltech Harvey Mudd Mathematics Competition

Team Round **February 20, 2010**

1. A matrix M is called *idempotent* if $M^2 = M$. Find an idempotent 2×2 matrix with distinct rational entries or write "none" if none exist.

2. The largest prime factor of $199^4 + 4$ has four digits. Compute the second largest prime factor.

3. Assume that the earth is a perfect sphere. A plane flies between $30°N\,45°W$ and $30°N\,45°E$ along the shortest possible route. Let θ be the northernmost latitude that the plane flies over. Compute $\sin\theta$.

4. Compute the number of integer solutions (x, y) to $xy - 18x - 35y = 1890$.

5. The *popularity* of a positive integer n is the number of positive integer divisors of n. For example, 1 has popularity 1, and 12 has popularity 6. For each number n between 1 and 30 inclusive, Cathy writes the number n on k pieces of paper, where k is the popularity of n. Cathy then picks a piece of paper at random. Compute the probability that she will pick an even integer.

6. Zach rolls five tetrahedral dice, each of whose faces are labeled 1, 2, 3, and 4. Compute the probability that the sum of the values of the faces that the dice land on is divisible by 3.

7. Compute all real numbers a such that the polynomial $x^4 + ax^3 + 1$ has exactly one real root.

8. Alice and Bob are going to play a game called extra tricky double rock paper scissors (ETDRPS). In ETDRPS, each player simultaneously selects *two* moves, one for his or her right hand, and one for his or her left hand. Whereas Alice can play rock, paper, or scissors, Bob is only allowed to play rock or scissors. After revealing their moves, the players compare right hands and left hands separately. Alice wins if she wins *strictly* more hands than Bob. Otherwise, Bob wins. For example, if Alice and Bob were to both play rock with their right hands and scissors with their left hands, then both hands would be tied, so Bob would win the game. However, if Alice were to instead play rock with both hands, then Alice would win the left hand. The right hand would still be tied, so Alice would win the game. Assuming both players play optimally, compute the probability that Alice will win the game.

9. Compute the positive integer n such that $\log_3 n < \log_2 3 < \log_3(n + 1)$.

10. Compute the number of 10-bit sequences of 0's and 1's do not contain 001 as a subsequence.