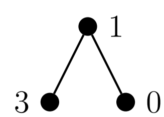
2010 Fall CHMMC Power Round

In this round, you will explore the *pebbling number* of graphs. For this part of the contest, you must fully justify all of your answers unless otherwise specfied. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems. Be sure to read the background information below before working on the problems.

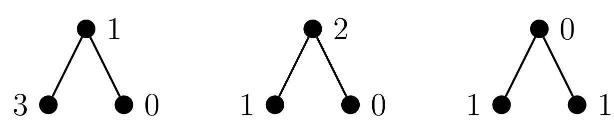
A graph G is a collection of vertices with some pairs of vertices linked by edges. For this problem, all the graphs are finite, all edges are undirected, there are no edges that go from a vertex to itself, and there cannot be more than one edge between two vertices. We say that a vertex u is a *neighbor* of a vertex v if there is an edge between u and v. A graph is *connected* if for any pair of vertices one can find a path from one to the other along the edges of the graph.

Alice and Bob play a game on a finite graph G. They have k pebbles, where k is a positive integer. Alice sets up the game by taking the k pebbles and placing them on the vertices of the graph, distributing them in any way she wishes. Alice then marks a vertex as the target vertex. Bob then tries to get at least one pebble onto the target vertex. However, Bob is only allowed to move pebbles as follows: If there is a vertex v with at least two pebbles on it, Bob can remove two pebbles from it and then place one of these pebbles on one of the neighbors of v. (The second pebble is removed from the game.) If, after moving pebbles in this manner, Bob manages to move a pebble onto the target vertex, then Bob wins. Otherwise, Alice wins. In particular, if Alice sets up the game with a pebble on the target vertex, then Bob wins automatically.

For example, if G is the graph below and $k = 4$, Alice can set up the game as follows:

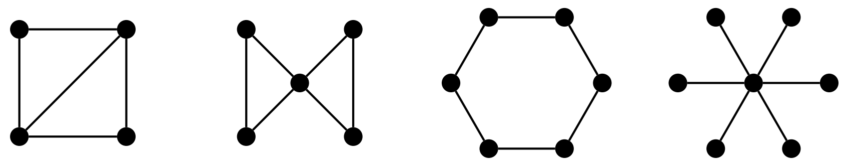


If Alice selects the rightmost vertex as the target vertex, then Bob can win by removing two pebbles from the leftmost vertex and placing a pebble on the middle vertex, and then removing two pebbles from the middle vertex and placing one pebble on the rightmost vertex, as shown below.



The *pebbling number* of a connected graph G, denoted $\pi(G)$, is defined to be the minimum positive integer k such that Bob can always win, no matter how Alice distributes the k pebbles or chooses the target vertex.

1. Give the pebbling number of the graphs illustrated below. Demonstrate an initial configuration with one fewer pebble for which Bob does not win. For this problem only, you do not need to prove your pebbling number is correct or that your configuration is unwinnable.



G_1 G_2 G_3 G_4

Solution: The pebbling number of G_1 is 4. Bob does not win if 1 pebble is placed on every vertex but the target one.

The pebbling number of G_2 is 6. Bob does not win if 1 pebble is placed on each of the top left and top right corner vertices, and 3 pebbles are placed on the bottom left corner, with the target vertex the bottom right.

The pebbling number of G_3 is 8. Bob does not win if 7 pebbles are placed on a vertex and the target vertex is the one distance 3 away.

The pebbling number of G_4 is 8. Bob does not win if the target vertex is the left one, 3 pebbles are placed in the right one, and 1 pebble is placed in each of the four vertices on the top and bottom.

2. Let K_n denote the complete graph, which is defined to be the graph containing n vertices with all possible edges drawn between them. Find the pebbling number of K_n for each positive integer $n \geq 2$.

Solution: We show $\pi(K_n) = n$. Let v be any vertex of K_n. If $n-1$ pebbles are used, Alice wins by placing no pebbles on v and 1 pebble on every other vertex and making v the target vertex. On the other hand, if n pebbles are used, then either every vertex has a pebble, in which case Bob has already won, or a vertex v has two pebbles. Then the pebbles on v can be moved to the target vertex, and Bob wins.

3. Suppose that G is a connected graph with n vertices. Show that $\pi(G) \geq n$.

Solution: It suffices to show Alice wins whenever $n-1$ pebbles are used. Let v be any vertex of G, and have Alice place one pebble on every vertex other than v, while making v the target vertex. Bob can make no moves, so he cannot win.

4. (a) Let P_n denote the n-path, a graph containing n vertices $v_1, v_2, \ldots v_n$ with an edge between v_i and v_{i+1} for $i = 1, 2, \ldots n-1$. Find $\pi(P_n)$ for each positive integer $n \geq 2$.

   (b) Let C_n denote the n-cycle, the same as P_n except there is an extra edge connecting v_1 and v_n. Show that $\pi(C_n) = 2^{n/2}$ for all *even* positive integers $n \geq 4$.

Solution:

(a) $\pi(P_n) = 2^{n-1}$. If less pebbles are used, Alice can place them all on v_1 and make v_n the target vertex. Then Bob by induction can get at most $2^{n-k} - 1$ pebbles onto v_k, which is not enough to get all the way to v_n.

To show Bob wins with 2^{n-1} pebbles, we proceed by induction. The base case $n = 1$ is trivial. Given it is true for $n-1$, consider a configuration of 2^{n-1} pebbles on P_n. WLOG the target vertex is not v_n, else flip the whole graph around so v_1 and v_n swap places. Consider vertex v_n, and suppose it has a pebbles. We can get $\lfloor a/2 \rfloor$ of them onto v_{n-1}. Then the total number of pebbles on $v_1, v_2, \dots v_{n-1}$ is

$$2^{n-1} - a + \lfloor a/2 \rfloor = 2^{n-1} - \lceil a/2 \rceil \geq 2^{n-1} - \lceil 2^{n-1}/2 \rceil = 2^{n-2}.$$

The last inequality follows from $a \leq 2^{n-1}$. So after moving as many pebbles as we can off v_n, we have 2^{n-2} pebbles on the other vertices. By the inductive hypothesis, we can reach any vertex since the remaining vertices form a copy of P_{n-1}. So Bob can always win.

(b) Suppose $2^{n/2} - 1$ pebbles were used. Alice can place them all on v_1 and make $v_{1+n/2}$ the target vertex. Then by induction Bob can get at most $2^k - 1$ pebbles onto $v_{1+n/2\pm k}$ for $1 \leq k \leq n/2 - 1$, which is not enough to make it to $v_{1+n/2}$.

To show Bob wins with $2^{n/2}$ pebbles, suppose WLOG that v_1 is the target vertex, and that vertex v_i has p_i pebbles for $2 \leq i \leq n$. Let

$$S_1 = \frac{p_2}{2} + \frac{p_3}{4} + \dots + \frac{p_n}{2^{n-1}},$$

$$S_2 = \frac{p_n}{2} + \frac{p_{n-1}}{4} + \dots + \frac{p_2}{2^{n-1}}.$$

Observe that $S_1 + S_2 \geq \sum_{i=1}^{n} \frac{p_i}{2^{n/2-1}} = 2$. So one of S_1, S_2 is greater than or equal to 1. WLOG $S_1 \geq 1$. Move as many pebbles as possible from v_n to v_{n-1}, so v_n has at most 1 pebble. This new configuration has the same value of S_1. Then move everything possible from v_{n-1} to v_{n-2}. Continue in this way until moving pebbles to v_2, meaning that $v_3, v_4, \dots v_n$ all have at most 1 pebble. Let $q_2, q_3, \dots q_n$ denote the new number of pebbles on each vertex. S_1 is still at least 1, and

$$\frac{q_3}{4} + \dots + \frac{q_n}{2^{n-1}} \leq \frac{1}{4} + \dots + \frac{1}{2^{n-1}} < \frac{1}{2}.$$

Therefore, $\frac{q_2}{2} > \frac{1}{2}$, so $q_2 \geq 2$. That means we have two pebbles on v_2, which Bob can move onto the target vertex v_1 and win.

5. Let G be a connected graph with n vertices. Let v be a vertex of G, and let H be the graph obtained by deleting v and all of its edges from G. Suppose that H is connected and that $\pi(H) = m$. Show that $\pi(G) \leq 2m$.

Solution: It suffices to show that Bob can always win if $2m$ pebbles are used. Suppose the target vertex is v, the one deleted to form H. If v has any pebbles Bob wins automatically. Otherwise, $2m$ pebbles are on H. Arbitrarily split the $2m$ pebbles into two sets A and B with m pebbles each. Let u be any neighbor of v (u exists since G is connected). Because $\pi(H) = m$, we can use the m pebbles in A to get a pebble to u. Likewise, we can use the m pebbles in B to get a second pebble to u. Now move the two pebbles from u to v and win.

Now suppose the target vertex is in H, and that v has a pebbles. Move as many pebbles from v onto any neighbor as possible; we can make $\lfloor a/2 \rfloor$ such moves. Then the total number of pebbles on H is

$$2m - a + \lfloor a/2 \rfloor = 2m - \lceil a/2 \rceil \geq 2m - \lceil 2m/2 \rceil = m.$$

So H has at least m pebbles, and since $\pi(H) = m$ we can reach any target vertex with them. So Bob wins in this case as well.

6. (a) Suppose that G is a connected graph with n vertices. Show that $\pi(G) \leq 2^{n-1}$.
   (b) Find all connected graphs G such that $\pi(G) = 2^{n-1}$.

   Solution:

   (a) We first prove a lemma: any connected graph G contains a vertex v that can be removed with all of its edges and still leave a connected graph. If G contains a cycle, cut an arbitrary edge of this cycle; this still leaves a connected graph. Repeat until no more cycles remain. Call this new graph G'; since it has no cycles, it is a tree. A tree contains at least one vertex u of degree 1. This can be seen by the fact that G' has one less edge than it has vertices, so if all vertices had degree at least two there would be too many edges. Therefore, if we remove u from G', all the other vertices will still be connected. Likewise, if we remove u from G, all other vertices will be connected by the edges present in G', which G contains. This proves the lemma.

   We prove the problem by induction on n, the number of vertices. The base case $n = 1$ is trivial. Suppose we have proven the case $n-1$, and let G be an arbitrary graph with n vertices. By the lemma, a vertex v of G exists that can be removed and still leave a connected graph. Let H be the graph obtained by removing v and all of its edges. By the inductive hypothesis, $\pi(H) \leq 2^{n-2}$. By problem 5, $\pi(G) \leq 2\pi(H) \leq 2^{n-1}$. This completes the induction.

   (b) TODO

7. Call a graph *tight* if for any pair of vertices u, v, either u and v are neighbors or there is a vertex w such that w is a neighbor of both u and v.

   (a) For each positive integer $n \geq 3$, give an example of a tight graph G with n vertices such that $\pi(G) = n + 1$.

   (b) Let G be a tight graph with n vertices. Show that $\pi(G) \leq 2n$.

Solution:

(a) Consider the graph G with vertices $v_1, v_2, \dots v_n$ such that v_1 is adjacent to all other vertices and no other edges exist. We claim $\pi(G) = n + 1$. First, we show Alice can win if n pebbles are used. Have her place 1 pebble on each of $v_4, \dots v_n$, 3 pebbles on v_3, and make v_2 the target vertex. The only move Bob can make is to use the two pebbles on v_3 to put one on v_1, and then he is stuck.

Now we show Bob wins if $n + 1$ pebbles are used. If v_1 is the target vertex, then either it has a pebble or some other vertex has at least 2 pebbles, so this is easy. Suppose WLOG that v_2 is the target vertex. If v_1 has at least 2 pebbles, Bob wins in one move. If v_1 has one pebble, at least one of $v_3, \dots v_n$ has 2 pebbles, so Bob moves from that vertex to v_1, then from v_1 to v_2 and wins. Otherwise, v_1 has no pebbles. If any of $v_3, \dots v_n$ have 4 pebbles, Bob moves two pebbles to v_1 and then wins. If at least two of $v_3, \dots v_n$ have at least 2 pebbles, Bob moves a pebble from each of them to v_1 and then wins. If neither of these is true, then out of $v_3, \dots v_n$ one vertex has at most 3 pebbles and all others have at most 1. This gives at most $3 + (n - 3) = n$ pebbles, a contradiction. So Bob wins in every case.

(b) It suffices to show $2n$ pebbles are enough to win. Suppose Alice marked v as the target vertex. Let $u_1, u_2, \dots u_d$ be the neighbors of v. Let $w_1, \dots w_{n-1-d}$ be the other vertices; each w_i is a neighbor of some u_i by the tight condition. Partition the w_i into $A_1, A_2, \dots A_d$, so that all vertices in A_i are neighbors of u_i. (Some w_i may be neighbors of multiple u_i; arbitrarily choose one.)

Notice that v, u_i and the vertices in A_i, together with just the edges between u_i and these vertices, form a star graph with $|A_i| + 2$ vertices. In problem 7a, we showed that this has pebbling number $|A_i| + 3$. So if at least $|A_i| + 3$ pebbles have been placed on these vertices, Bob will win. Suppose then that $|A_i| + 2$ pebbles at most have been placed on them.

Consider the total number of pebbles on all vertices. This is the sum of all the pebbles on all the u_i and A_i. By the above paragraph, this is at most $\sum_{i=1}^{d}(2 + |A_i|) = 2d + (n - d - 1) = n + d - 1$. Since d is the degree of v, $d \leq n - 1$. So at most $2n - 2$ pebbles have been placed, a contradiction. Some u_i, A_i together have at least $|A_i| + 3$ pebbles, and Bob will win.