Caltech Harvey Mudd Mathematics Competition

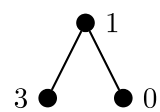
Power Round **November 13, 2010**

In this round, you will explore the *pebbling number* of graphs. For this part of the contest, you must fully justify all of your answers unless otherwise specfied. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems. Be sure to read the background information below before working on the problems.

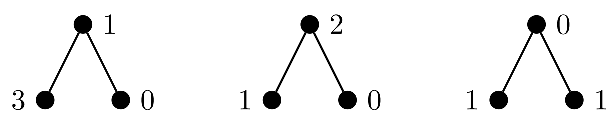
A graph G is a collection of vertices with some pairs of vertices linked by edges. For this problem, all the graphs are finite, all edges are undirected, there are no edges that go from a vertex to itself, and there cannot be more than one edge between two vertices. We say that a vertex u is a *neighbor* of a vertex v if there is an edge between u and v. A graph is *connected* if for any pair of vertices one can find a path from one to the other along the edges of the graph.

Alice and Bob play a game on a finite graph G. They have k pebbles, where k is a positive integer. Alice sets up the game by taking the k pebbles and placing them on the vertices of the graph, distributing them in any way she wishes. Alice then marks a vertex as the target vertex. Bob then tries to get at least one pebble onto the target vertex. However, Bob is only allowed to move pebbles as follows: If there is a vertex v with at least two pebbles on it, Bob can remove two pebbles from it and then place one of these pebbles on one of the neighbors of v. (The second pebble is removed from the game.) If, after moving pebbles in this manner, Bob manages to move a pebble onto the target vertex, then Bob wins. Otherwise, Alice wins. In particular, if Alice sets up the game with a pebble on the target vertex, then Bob wins automatically.

For example, if G is the graph below and $k = 4$, Alice can set up the game as follows:

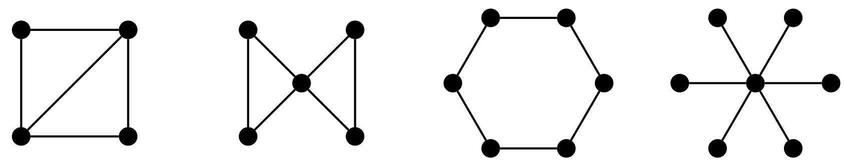


If Alice selects the rightmost vertex as the target vertex, then Bob can win by removing two pebbles from the leftmost vertex and placing a pebble on the middle vertex, and then removing two pebbles from the middle vertex and placing one pebble on the rightmost vertex, as shown below.



The *pebbling number* of a connected graph G, denoted $\pi(G)$, is defined to be the minimum positive integer k such that Bob can always win, no matter how Alice distributes the k pebbles or chooses the target vertex.

1. Give the pebbling number of the graphs illustrated below. Demonstrate an initial configuration with one fewer pebble for which Bob does not win. For this problem only, you do not need to prove your pebbling number is correct or that your configuration is unwinnable.



G_1 G_2 G_3 G_4

2. Let K_n denote the complete graph, which is defined to be the graph containing n vertices with all possible edges drawn between them. Find the pebbling number of K_n for each positive integer $n \geq 2$.

3. Suppose that G is a connected graph with n vertices. Show that $\pi(G) \geq n$.

4. (a) Let P_n denote the n-path, a graph containing n vertices $v_1, v_2, \dots v_n$ with an edge between v_i and v_{i+1} for $i = 1, 2, \dots n-1$. Find $\pi(P_n)$ for each positive integer $n \geq 2$.

   (b) Let C_n denote the n-cycle, the same as P_n except there is an extra edge connecting v_1 and v_n. Show that $\pi(C_n) = 2^{n/2}$ for all *even* positive integers $n \geq 4$.

5. Let G be a connected graph with n vertices. Let v be a vertex of G, and let H be the graph obtained by deleting v and all of its edges from G. Suppose that H is connected and that $\pi(H) = m$. Show that $\pi(G) \leq 2m$.

6. (a) Suppose that G is a connected graph with n vertices. Show that $\pi(G) \leq 2^{n-1}$.

   (b) Find all connected graphs G such that $\pi(G) = 2^{n-1}$.

7. Call a graph *tight* if for any pair of vertices u, v, either u and v are neighbors or there is a vertex w such that w is a neighbor of both u and v.

   (a) For each positive integer $n \geq 3$, give an example of a tight graph G with n vertices such that $\pi(G) = n + 1$.

   (b) Let G be a tight graph with n vertices. Show that $\pi(G) \leq 2n$.