Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Individual Round

The individual round will last for an hour. The test will have fifteen questions worth 2 points each with numeric or algebraic answers. Like in the short answer part of the team round, students will not need to justify their answers to the individual round questions.

IR1. How many nonzero digits are in the number $(5^{94}+5^{92})(2^{94}+2^{92})$?

IR2. Suppose A is a set of 2013 distinct positive integers such that the arithmetic mean of any subset of A is also an integer. Find an example of A.

IR3. How many minutes until the smaller angle formed by the minute and hour hands on the face of a clock is congruent to the smaller angle between the hands at 5:15pm? Round your answer to the nearest minute.

IR4. Suppose a and b are positive real numbers, $a+b=1$, and

$$1+\frac{a^2+3b^2}{2ab}=\sqrt{4+\frac{a}{b}+\frac{3b}{a}}.$$

Find a.

IR5. Suppose $f(x)=\dfrac{e^x-12e^{-x}}{2}$. Find all x such that $f(x)=2$.

IR6. Let $P_1, P_2, \ldots, P_n$ be points equally spaced on a unit circle. For how many integer $n \in \{2, 3, \ldots, 2013\}$ is the product of all pairwise distances:

$$\prod_{1\le i<j\le n} P_iP_j$$

a rational number? Note that $\prod$ means the product. For example, $\prod_{1\le i\le 3} i = 1\cdot 2\cdot 3 = 6$.

IR7. Determine the value a such that the following sum converges if and only if $r\in(-\infty, a)$:

$$\sum_{n=1}^{\infty}(\sqrt{n^4+n^r}-n^2).$$

Note that $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges if and only if $s>1$.

IR8. Find two pairs of positive integers (a, b) with $a>b$ such that

$$a^2+b^2=40501.$$

IR9. Consider a simplified memory-knowledge model. Suppose your total knowledge level the night before you went to a college was 100 units. Each day, when you woke up in the morning you forgot 1% of what you had learned. Then, by going to lectures, working on the homework, preparing for presentations, you had learned more and so your knowledge level went up by 10 units at the end of the day.

According to this model, how long do you need to stay in college until you reach the knowledge level of exactly 1000?

IR10. Suppose $P(x) = 2x^8 + x^6 - x^4 + 1$, and that P has roots $a_1, a_2, \cdots, a_8$ (a complex number z is a root of the polynomial $P(x)$ if $P(z) = 0$). Find the value of

$$(a_1^2 - 2)(a_2^2 - 2)(a_3^2 - 2) \cdots (a_8^2 - 2).$$

IR11. Find all values of x satisfying $(x^2 + 2x - 5)^2 = -2x^2 - 3x + 15$.

IR12. Suppose x, y and z are *positive* real numbers such that

$$x^2 + y^2 + xy = 9,$$
$$y^2 + z^2 + yz = 16,$$
$$x^2 + z^2 + xz = 25.$$

Find $xy + yz + xz$ (the answer is unique).

IR13. Suppose that $P(x)$ is a monic polynomial (i.e, the leading coefficient is 1) with 20 roots, each distinct and of the form $\frac{1}{3^k}$ for $k = 0, 1, 2, \ldots, 19$. Find the coefficient of x^{18} in $P(x)$.

IR14. Find the sum of the reciprocals of all perfect squares whose prime factorization contains only powers of $3, 5, 7$ (i.e. $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{225} + \frac{1}{441} + \frac{1}{625} + \cdots$).

IR15. Find the number of integer quadruples (a, b, c, d) which also satisfy the following system of equations:

$$1 + b + c^2 + d^3 = 0,$$
$$a + b^2 + c^3 + d^4 = 0,$$
$$a^2 + b^3 + c^4 + d^5 = 0,$$
$$a^3 + b^4 + c^5 + d^6 = 0.$$