Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Power Round

In this round you will prove an identity from both algebraic and combinatorial perspectives. For this part of the contest, you must fully justify all your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems.

PR1. (a) Evaluate

$$\left(\frac{2-1}{2}\right) * \left(1 + \left(2^{-1}\frac{2^1}{2^1-1}\right)\right).$$

(b) Define $\sum_{i=a}^{b} k_i$ to be the sum of numbers k_i when i ranges from a to b. For instance, $\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2 = 14$ Similarly, define $\prod_{i=a}^{b} k_i$ to be a product of numbers. For example, $\prod_{i=1}^{5} i = 1 * 2 * 3 * 4 * 5 = 120$

Now, let

$$S_n = \left(\prod_{i=1}^{n} \frac{2^i - 1}{2^i}\right) * \left(1 + \sum_{i=1}^{n}\left(2^{-i} * \prod_{j=1}^{i} \frac{2^j}{2^j - 1}\right)\right).$$

Find a recurrence relation, writing S_{n+1} in terms of S_n.

(c) Given that

$$\prod_{i=1}^{\infty} \frac{2^i - 1}{2^i} \neq 0.$$

Show by induction that

$$\left(\prod_{i=1}^{\infty} \frac{2^i - 1}{2^i}\right) * \left(1 + \sum_{i=1}^{\infty}\left(2^{-i} * \prod_{j=1}^{i} \frac{2^j}{2^j - 1}\right)\right) = 1.$$

PR2. (a) A combinatorial proof is one that count the same value in two ways. For instance, here is a combinatorial proof that $1/2 + 1/4 + 1/8 + 1/16 + \ldots = 1$:

Consider someone flipping a coin infinitely many times. There is a probability of 1 that a tails shows up in this sequence of flips. Consider the various possible cases of this; the probability that the first tails happens at the nth toss. The chance that it happens at the first toss is $1/2$. The chance that it happens at the second toss is $1/4$, since we must start with the sequence HT where H represents a heads and T represents a tails. Similarly, the chance that the 3rd toss is the first tails is $1/8$, and so on. Adding up these possibilities, we get that a total chance of $1/2 + 1/4 + 1/8 + 1/16 + \ldots$ that a tails occurs eventually. Therefore, $1/2 + 1/4 + 1/8 + 1/16 + \ldots = 1$.

Define $\binom{n}{k}$ as the number of ways to choose k things out of n choices. For instance, $\binom{4}{2} = 6$ since, out of 4 options (say a, b, c, d), there are 6 ways to pick 2 of them: $((a, b)$, or (a, c), or (a, d), or (b, c), or (b, d), or $(c, d))$. Give a combinatorial proof (without any algebra) that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(b) Given that, for any non-negative integer n, we have that the sum, for all possible combinations of distinct positive integers $k_1, k_2, \ldots k_n$,

$$\sum_{k_1, k_2, \ldots k_n} \frac{1}{2^{k_i} - 1} = 2^{-n} * \prod_{j=1}^{n} \frac{2^j}{2^j - 1}.$$

Provide a combinatorial proof that

$$\left(\prod_{i=1}^{\infty} \frac{2^i - 1}{2^i}\right) * \left(1 + \sum_{i=1}^{\infty} \left(2^{-i} * \prod_{j=1}^{i} \frac{2^j}{2^j - 1}\right)\right) = 1.$$