Fall 2012 Caltech-Harvey Mudd Math Competition

November 17, 2012

Team Round Solutions

The team round will last for **75 minutes**, plus a five minute reading period at the beginning. The test will have two equally weighted parts: a power question and a short answer part. Teams will be able to work on the two parts simultaneously.

- During the five minute reading period, team members may not write. However, they may discuss the problems with each other.
- Teams may use blackboards or whiteboards.
- The power question will be similar in style to the ARML power round, although it will be shorter. Teams will be expected to fully justify their answers to the power question. The power question will be worth 90 points.
- The short answer part of the team round will have ten questions worth 9 points each with numerical or algebraic answers. In the short answer part, teams will not need to justify their answers. To get full credit, they will need only to write down the correct answer.

TR1. 1

TR2. $\frac{5}{2} = 2.5$.

TR3. 3,3,4,4,5,5 and 3,3,3,5,5,5 (the order does not matter)

TR4. $2\sqrt{2}$

TR5. 17/81

TR6. 4

TR7. $3\sqrt{2}$

TR8. $5 \times 10^{49} + 5 \times 10^{47} - 10^{20}$

TR9. 1536

TR10. 2013

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Power Round Solutions

In this round you will prove an identity from both algebraic and combinatorial perspectives. For this part of the contest, you must fully justify all your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems.

PR1. (a) We get

$$\frac{1}{2} * (1+1) = 1$$

(b) We have

$$\begin{aligned} S_{n+1} &= \left(\prod_{i=1}^{n+1} \frac{2^i-1}{2^i}\right) * \left(1 + \sum_{i=1}^{n+1} \left(2^{-i} * \prod_{j=1}^{i} \frac{2^j}{2^j-1}\right)\right) \\ &= \frac{2^{n+1}-1}{2^{n+1}} \left(\prod_{i=1}^{n} \frac{2^i-1}{2^i}\right) * \left(1 + \sum_{i=1}^{n} \left(2^{-i} * \prod_{j=1}^{i} \frac{2^j}{2^j-1}\right) + 2^{-n-1} * \prod_{j=1}^{n+1} \frac{2^j}{2^j-1}\right) \\ &= \frac{2^{n+1}-1}{2^{n+1}} S_n + \left(\prod_{i=1}^{n+1} \frac{2^i-1}{2^i}\right) * \left(2^{-n-1} * \prod_{j=1}^{n+1} \frac{2^j}{2^j-1}\right). \end{aligned}$$

(c) We will induct on n. Our base case is S_1, which we proved to be 1 earlier. Now we know S_{n+1} to be

$$S_{n+1} = \frac{2^{n+1}-1}{2^{n+1}} S_n + \left(\prod_{i=1}^{n+1} \frac{2^i-1}{2^i}\right) * \left(2^{-n-1} * \prod_{j=1}^{n+1} \frac{2^j}{2^j-1}\right).$$

So make the inductive hypothesis that $S_n = 1$. Then

$$\begin{aligned} S_{n+1} &= \frac{2^{n+1}-1}{2^{n+1}} + \left(\prod_{i=1}^{n+1} \frac{2^i-1}{2^i}\right) * \left(2^{-n-1} * \prod_{j=1}^{n+1} \frac{2^j}{2^j-1}\right) \\ &= \frac{2^{n+1}-1}{2^{n+1}} + \left(2^{-n-1}\right) \\ &= 1 - \frac{1}{2^{n+1}} + 2^{-n-1} \\ &= 1. \end{aligned}$$

Which proves the $n+1$th case. So, by the principle of induction, $S_n = 1$ for all n. We are given that

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{2^i-1}{2^i}$$

exists and is non-zero, so

$$\lim_{n\to\infty}\left(\prod_{i=1}^{n}\frac{2^i-1}{2^i}\right)*\left(1+\sum_{i=1}^{n}\left(2^{-i}*\prod_{j=1}^{i}\frac{2^j}{2^j-1}\right)\right)=\left(\prod_{i=1}^{\infty}\frac{2^i-1}{2^i}\right)*\left(1+\sum_{i=1}^{\infty}\left(2^{-i}*\prod_{j=1}^{i}\frac{2^j}{2^j-1}\right)\right).$$

And hence

$$\left(\prod_{i=1}^{\infty}\frac{2^i-1}{2^i}\right)*\left(1+\sum_{i=1}^{\infty}\left(2^{-i}*\prod_{j=1}^{i}\frac{2^j}{2^j-1}\right)\right)=1.$$

PR2. (a) Consider two cases: one in which the first element is one of the k elements picked, and one in which it is not. If it is picked, then there are $k-1$ elements left to pick among $n-1$ elements left; this gives $\binom{n-1}{k-1}$ possibilities. If the first element is not picked, then there are k elements left to pick among $n-1$ elements left; this gives $\binom{n-1}{k}$ possibilities. Hence $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$.

(b) Consider the following scenario:

There are infinitely many zebras trying to cross lion-infested areas to get to an oasis. They travel one at a time,and the first has a 1/2 chance of making it across. If any zebra dies, the number of lions doubles, so all future zebras have only 1/2 the chance of making it across. However, if one survives, the chance of survival stays the same. For instance, the chance that the first zebra survives, the second dies, and the third survives is $1/2 * 1/2 * 1/4 = 1/16$. Then the probability that some number of zebras makes it across is 1, and the probability that n zebras make is the sum over all possible combinations of distinct positive integers $k_1, k_2, \ldots k_n$ of $\prod_{i=1}^{\infty}\frac{2^i-1}{2^i}*\frac{1}{2^{k_i}-1}$. By our given, this is

$$\prod_{i=1}^{\infty}\frac{2^i-1}{2^i}*2^{-n}*\prod_{j=1}^{n}\frac{2^j}{2^j-1}$$

Hence, the sum over all possible n is

$$\left(\prod_{i=1}^{\infty}\frac{2^i-1}{2^i}\right)*\left(1+\sum_{i=1}^{\infty}\left(2^{-i}*\prod_{j=1}^{i}\frac{2^j}{2^j-1}\right)\right)$$

which must just be 1.

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Individual Round Solutions

The individual round will last for an hour. The test will have fifteen questions worth 2 points each with numeric or algebraic answers. Like in the short answer part of the team round, students will not need to justify their answers to the individual round questions.

IR1. 2

IR2. $A = \{2013! * 1, 2013! * 2 \ldots, 2013! * 2013\}$. Other examples should be easy to check.

IR3. 25 minutes

IR4. $\frac{3-\sqrt{3}}{2}$

IR5. $\ln 6$

IR6. 21

IR7. 2

IR8. $(201, 10)$ and $(199, 30)$

IR9. forever (or your knowledge level will never reach 1000)

IR10. $\left(\frac{37}{2}\right)^2 = \frac{1369}{4}$.

IR11. $x = \frac{-3\pm\sqrt{17}}{2}, \frac{-1\pm\sqrt{21}}{2}$

IR12. $8\sqrt{3}$

IR13. $\frac{9}{16}(1-3^{-19})(1-3^{-20})$

IR14. $\frac{9216}{11025}$

IR15. 3

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Mixer Round Solutions

In the mixer round, students will be grouped with students from other teams. This round will not count towards the students' or teams' scores, but there will be a separate small prize for the winning team of the mixer round.

MR1. We divide by 5^x both sides to get

$$(3/5)^x + (4/5)^x = 1.$$

Since the left hand side is a decreasing function in terms of x, there is only one solution. Any value of x higher than 2 make the left side smaller than 1, while any value of x lower than 2 will make the left side larger than 1.

MR2. The answer is 4. Let $a = \sqrt{9+4\sqrt{5}}$ and $b = \sqrt{9-4\sqrt{5}}$. Hence, we have

$$(a-b)^2 = a^2 - 2ab + b^2 = 16.$$

Since $a > b$, we must have that $a - b = \sqrt{9+4\sqrt{5}} - \sqrt{9-4\sqrt{5}} = 4$.

MR3. There is a winning strategy for A. First round, put the coin at the center of the table.Then, each time, if B can place a coin somewhere, there will be a guaranteed spot on the opposite side of the table (with respect to the center)because of the symmetry. If A keeps doing this,eventually B will run out of space!

MR4. The answer is 0. If there is a way to make it bigger, there must also be a way to make it smaller, since every move is reversible. Initially, if we place the 4 pegs on the lattice points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, there is no way to move outside the lattice points. Hence we can see there is no way to make the arrangement smaller than the original one.

MR5. 173

MR6. 2

MR7. $2^{98} - 2^{49}$

MR8. 4036087

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Tiebreaker Round Solutions

The tiebreaker round will be similar to the ARML tiebreaker round. The top students with the same score on the individual round will be given a question to be solved in ten minutes. The students will be able to submit an answer only once, and they will be ranked according to the time when they submit a correct answer.

TBR1. 7

TBR2. There are two solutions: $f(x) = \left(\frac{1+\sqrt{5}}{2}\right) x$ for all x, and $f(x) = \left(\frac{1-\sqrt{5}}{2}\right) x$ for all x.

TBR3. 26

TBR4. $i = k = 1, s = 3$.