Caltech Harvey Mudd Mathematics Competition

Individual Round **March 3, 2012**

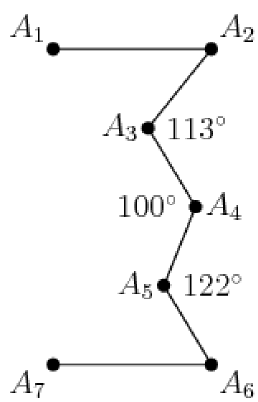
1. A robot is at position 0 on a number line. Each second, it randomly moves either one unit in the positive direction or one unit in the negative direction, with probability $\frac{1}{2}$ of doing each. Find the probability that after 4 seconds, the robot has returned to position 0.

   Solution: The answer is $\frac{3}{8}$. The robot has to make two moves forward and two moves backward to return to the start. There are $\binom{4}{2}$ ways to do this. Each sequence has probability $\frac{1}{2^4}$ of occurring, so the answer is $\frac{\binom{4}{2}}{2^4} = \frac{6}{16} = \frac{3}{8}$.

2. How many positive integers $n \le 20$ are such that the greatest common divisor of n and 20 is a prime number?

   Solution: The answer is 6. The possible prime values for $\gcd(n, 20)$ are $2, 5$. Since $\gcd(n, k) = d$ is equivalent to $\gcd(n/d, k/d) = 1$, there are $\phi(\frac{k}{d})$ solutions in n to $\gcd(n, k) = d$. Therefore, the total number of solutions is $\phi(\frac{20}{2}) + \phi(\frac{20}{5}) = 4 + 2 = 6$. (Alternatively, one can just list all the numbers from 1 to 20 and see which ones work.)

3. A sequence of points $A_1, A_2, A_3, \ldots A_7$ is shown in the diagram below, with A_1A_2 parallel to A_6A_7. We have $\angle A_2A_3A_4 = 113°$, $\angle A_3A_4A_5 = 100°$, and $\angle A_4A_5A_6 = 122°$. Find the degree measure of $\angle A_1A_2A_3 + \angle A_5A_6A_7$.



   Solution: The answer is 135°. If we draw the line connecting A_1 and A_7, we get a heptagon. Furthermore the interior angles at A_1 and A_7 sum to 180 degrees due to the parallel segments A_1A_2, A_6A_7. So the remaining 5 angles have to sum to 720 degrees. The interior angles at $\angle A_3, \angle A_4, \angle A_5$ have degree measures $360 - 113, 100, 360 - 122$, which sums to $720 - 135$. Therefore, the interior angles at A_2 and A_6 must have measures that sum to 135°.

4. Compute

$$\log_3\left(\frac{\log_3 3^{3^{3^3}}}{\log_{3^3} 3^{3^3}}\right).$$

   Solution: The answer is 25. Note that $\log_3 3^{3^{3^3}} = 3^{3^3} = 3^{27}$ and $\log_{3^3} 3^{3^3} = \frac{1}{3}\log_3 3^{3^3} = \frac{27}{3} = 3^2$. So the answer is $\log_3\left(\frac{3^{27}}{3^2}\right) = 27 - 2 = 25$.

5. In an 8×8 chessboard, a pawn has been placed on the third column and fourth row, and all the other squares are empty. It is possible to place nine rooks on this board such that no two rooks attack each other. How many ways can this be done? (Recall that a rook can attack any square in its row or column provided all the squares in between are empty.)

Solution: The answer is 14400. The third column and fourth row each need two rooks each to have nine rooks total, with one rook attacking the pawn from each of the four sides. The rook to the left has two options, the rook to the right has five options, the rook above has four options, and the rook below has three options, for $5 \cdot 4 \cdot 3 \cdot 2 = 120$. Regardless of where these four rooks go, placing the remaining five rooks is just like placing five rooks in a 5×5 board. There are $5! = 120$ ways to do this. So the answer is $120 \cdot 120 = 14400$.

6. Suppose that a, b are positive real numbers with $a > b$ and $ab = 8$. Find the minimum value of $\frac{a^2+b^2}{a-b}$.

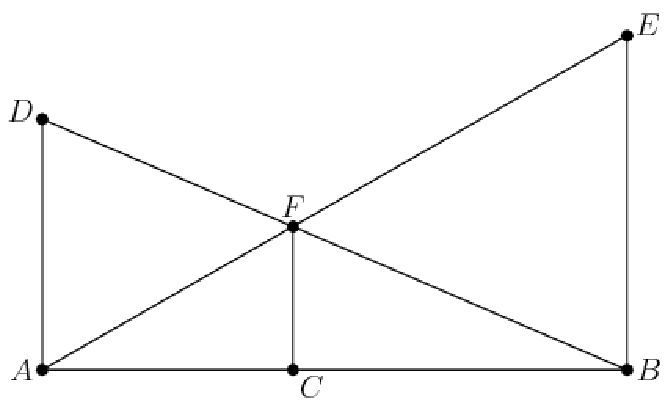
   Solution: The answer is 8. Rewrite the fraction as

$$\frac{a^2+b^2}{a-b} = \frac{a^2+b^2-2ab+16}{a-b} = \frac{(a-b)^2}{a-b} + \frac{16}{a-b} = (a-b) + \frac{16}{a-b}.$$

   Applying AM-GM to these two terms, we get $(a-b) + \frac{16}{a-b} \geq 8$, where equality is achieved when $a - b = 4$. This can be achieved by setting $a = 2\sqrt{3}+2$ and $b = 2\sqrt{3}-2$ (these values are obtained by solving the quadratic $x^2 - 4x - 8 = 0$).

7. A cone of radius 4 and height 7 has A as its apex and B as the center of its base. A second cone of radius 3 and height 7 has B as its apex and A as the center of its base. What is the volume of the region contained in both cones?

   Solution: The answer is $\frac{48\pi}{7}$. The surfaces of the two cones intersect at a circle O coplanar with each cone's base; let the center of O be C. Consider the cross-section formed by any plane passing through A, B, C.



   The diagram above shows half of the intersection of the plane and the two cones (the other half is the reflection about AB). Let D be the point on this plane such that AD is a radius of the base of the smaller cone, and similarly define E for the larger cone. Let F be the intersection of DB and EA, meaning FC is a radius of O. Because AD, BE, CF are parallel, we have $\triangle ADB \sim \triangle CFB$ and $\triangle BEA \sim \triangle CFA$. So $\frac{FC}{AC} = \frac{EB}{AB} = \frac{4}{7}$ and $\frac{FC}{BC} = \frac{DA}{BA} = \frac{3}{7}$. Therefore $4AC = 3BC = 3(7 - AC)$, so $AC = 3$ and $FC = \frac{12}{7}$, the radius of O.

   To find the common volume, we partition the common intersection into two regions, one region on one side of the plane containing O and the other region on the other side. Each region is a cone with base O and apex either A or B. Let the height of one cone be h. Then the volume of that cone is $\frac{\pi}{3}\left(\frac{12}{7}\right)^2 h$ and the volume of the other is $\frac{\pi}{3}\left(\frac{12}{7}\right)^2 (7-h)$. Taking their sum gives us a total volume of $\frac{48\pi}{7}$.

8. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be a permutation of the numbers $1, 2, 3, 4, 5, 6$. We say a_i is *visible* if a_i is greater than any number that comes before it; that is, $a_j < a_i$ for all $j < i$. For example, the permutation $2, 4, 1, 3, 6, 5$ has three visible elements: $2, 4, 6$. How many such permutations have exactly two visible elements?

   Solution: The answer is 274. Suppose the first element of the permutation is k. If $k = 6$, then it is the only visible element, so there are no solutions here. Otherwise, the two visible elements will be $k, 6$, so we only need to require that all elements strictly between these two appear after the 6. There are $5 - k$ such elements $(k+1, k+2, \dots 5)$, and if we restrict the permutation to just 6 and those $5-k$ elements, a proportion of $\frac{1}{6-k}$ of those permutations have the 6 in front. So of the 5! permutations with $k < 6$ as the first element, $\frac{5!}{6-k}$ of them have two visible elements. So the answer is

$$\sum_{k=1}^{5} \frac{5!}{(6-k)!} = 5!\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = 120 + 60 + 40 + 30 + 24 = 274.$$

9. Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$, and let $S = [f(6)]^5 + [f(10)]^3 + [f(15)]^2$. Compute the remainder when S is divided by 30.

   Solution: The answer is 21. Notice that x is a divisor of $f(x)$, so $f(k)$ is divisible by k for all k. So $[f(6)]^5 \equiv 0 \pmod 6$, $[f(10)]^3 \equiv 0 \pmod{10}$, and $[f(15)]^2 \equiv 0 \pmod{15}$. By the fact that $f(1) = 21$ and Fermat's Little Theorem,

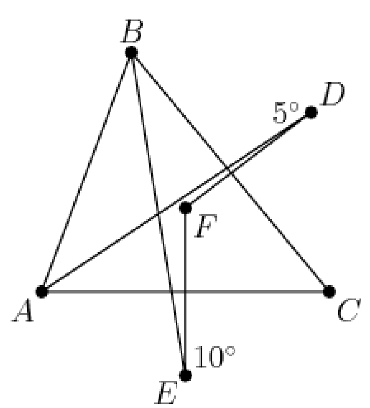
$$[f(6)]^5 \equiv f(6) \equiv f(1) \equiv 21 \pmod 5.$$

$$[f(10)]^3 \equiv f(10) \equiv f(1) \equiv 21 \pmod 3.$$

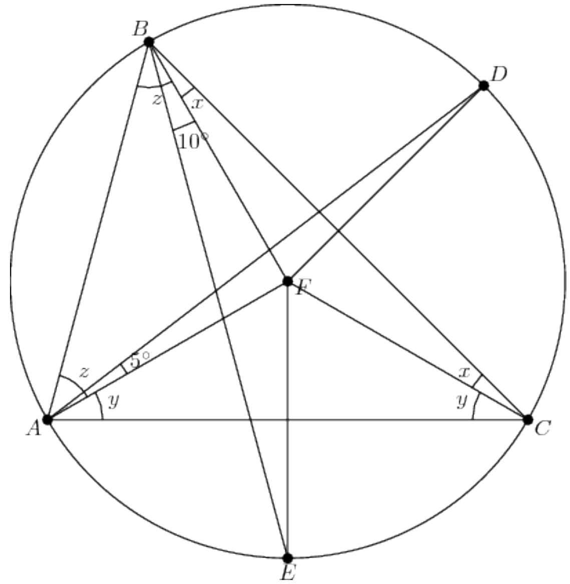
$$[f(15)]^2 \equiv f(15) \equiv f(1) \equiv 21 \pmod 2.$$

   So when we take the sum S, we get that S is equivalent to 21 mod 2, mod 3, and mod 5. By the Chinese Remainder Theorem, $S \equiv 21 \pmod{30}$.]

10. In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the length of DF.



Solution: The answer is $\sqrt{3}$. Notice that F is the circumcenter of triangle ABC, that D is the midpoint of arc BC on the circumcircle, and E is the midpoint of arc AC. Hence FA, FB, FC, FD, FE are all radii of the circumcircle and equal. This makes $\triangle ADF, \triangle BEF$ isosceles, so $\angle ADF = \angle DAF = 5°$ and $\angle BEF = \angle EBF = 10°$.



Let $\angle FBC = \angle FCB = x$, $\angle FCA = \angle FAC = y$, $\angle FAB = \angle FBA = z$. Using the various angles we know we can find that $x+y+z = 90°$, $y+5° = z-5°$, and $x+10° = z-10°$. Solving this system gives $(x, y, z) = (20°, 30°, 40°)$, so in particular $\angle ABC = 60°$. By the extended law of sines, $\frac{AC}{\sin\angle ABC} = 2R = 2 \cdot DF$. Therefore, $DF = \frac{3}{2\sin 60°} = \sqrt{3}$.

11. Let $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. How many subsets S of $\{1, 2, \dots 2011\}$ are there such that

$$F_{2012} - 1 = \sum_{i \in S} F_i?$$

Solution: The answer is 1006. We show that $F_{2n} - 1$ can be written as a sum of distinct Fibonacci numbers in n ways by induction. The base case $n = 1$ is obvious since only the empty set works.

Suppose it is true for n. For representing $F_{2n+2} - 1$, there are two cases: either we use F_{2n+1} or we do not. If we do use it, then the remaining terms must sum to $F_{2n} - 1$, and we know there are n ways to do this. Furthermore none of the ways will overlap with the F_{2n+1} term we already have since $F_{2n} - 1 < F_{2n+1}$. On the other hand, ifwe do not use the F_{2n+1} term, then the following Fibonacci identity,

$$F_{2n+2} - 1 = \sum_{k=1}^{2n} F_k,$$

shows that the only way to get the sum high enough is to use every term. Therefore there is one additional way, for $n + 1$ total, and the induction is complete. The answer to the given question is thus $\frac{2012}{2} = 1006$.

12. Let a_k be the number of perfect squares m such that $k^3 \le m < (k+1)^3$. For example, $a_2 = 3$ since three squares m satisfy $2^3 \le m < 3^3$, namely $9, 16$, and 25. Compute

$$\sum_{k=0}^{99} \lfloor \sqrt{k} \rfloor a_k,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Solution: The answer is 6975. Define

$$b_s = \sum_{k=s^2}^{(s+1)^2-1} \lfloor \sqrt{k} \rfloor a_k = s \sum_{k=s^2}^{(s+1)^2-1} a_k,$$

for $s = 0, 1, \ldots 10$. Note that $\sum_{k=s^2}^{(s+1)^2-1} a_k$ is the same as the number of perfect squares m between s^6 and $(s+1)^6$ by plugging into the definition. Since $s^6 = (s^3)^2$ and $(s+1)^6 = [(s+1)^3]^2$, there are $(s+1)^3 - s^3$ squares in between. So $b_s = s[(s+1)^3 - s^3]$. Then

$$\begin{aligned} \sum_{k=0}^{99} \lfloor \sqrt{k} \rfloor a_k &= \sum_{s=0}^{9} s[(s+1)^3 - s^3], \\ &= \sum_{s=0}^{9} \sum_{t=1}^{s} \left[(s+1)^3 - s^3\right], \end{aligned}$$

Switching the order of summation, this is equal to

$$\sum_{t=1}^{9} \sum_{s=t}^{9} \left[(s+1)^3 - s^3\right] = \sum_{t=1}^{9} \left(10^3 - t^3\right) = 9000 - \sum_{t=1}^{9} t^3,$$

so the answer is $9000 - \left(\frac{9 \cdot 10}{2}\right)^2 = 9000 - 2025 = 6975$.

13. Suppose that a, b, c, d, e, f are real numbers such that

$$a + b + c + d + e + f = 0,$$

$$a + 2b + 3c + 4d + 2e + 2f = 0,$$

$$a + 3b + 6c + 9d + 4e + 6f = 0,$$

$$a + 4b + 10c + 16d + 8e + 24f = 0,$$

$$a + 5b + 15c + 25d + 16e + 120f = 42,$$

Compute $a + 6b + 21c + 36d + 32e + 720f$.

Solution: The answer is 508. Let $g(k) = a + kb + \frac{k(k-1)}{2}c + k^2 d + 2^{k-1}e + k! \cdot f$. We are given $g(1), g(2), g(3), g(4), g(5)$ and we want to find $g(6)$. Because the coefficients of a, b, c, d are polynomials of degree at most 2, $g(k) - 3g(k-1) + 3g(k-2) - g(k-3)$ will have a coefficient of 0 for all of a, b, c, d. We use this observation to obtain the following three equations in three unknowns:

$$g(4) - 3g(3) + 3g(2) - g(1) = 0 = e + 11f,$$

$$g(5) - 3g(4) + 3g(3) - g(2) = 42 = 2e + 64f,$$

$$g(6) - 3g(5) + 3g(4) - g(3) = g(6) - 126 = 4e + 426f.$$

The first two equations allow us to determine that $f = 1$ and $e = -11$. Plugging this into the last one gives us $g(6) = 508$.

14. In Cartesian space, three spheres centered at $(-2, 5, 4), (2, 1, 4)$, and $(4, 7, 5)$ are all tangent to the xy-plane. The xy-plane is one of two planes tangent to all three spheres; the second plane can be written as the equation $ax + by + cz = d$ for some real numbers a, b, c, d. Find $\frac{c}{a}$.

    Solution: The answer is $-\frac{31}{8}$. First of all, translate the spheres 2 units in the positive x direction and 1 unit in the negative y direction to get spheres centered at $(0, 4, 4), (4, 0, 4), (6, 6, 5)$. This only changes d in the equation of the plane we are finding, not the normal vector, so the answer stays the same. We now compute the equation of the plane P passing through the centers of the spheres. If P has equation $a'x+b'y+c'z = 1$, then it satisfies system $4b'+4c' = 1$, $4a' + 4c' = 1$, and $6a' + 6b' + 5c' = 1$. Solving this gives $c' = \frac{2}{7}$ and $a' = b' = -\frac{1}{28}$.

    Notice now that the spheres are symmetric about P. In particular, if we reflect the xy-plane over P, we will get the second plane tangent to all of the spheres. The normal vector of this new plane will be the normal vector of the xy-plane reflected about the normal vector of P. These normal vectors are $(0, 0, 1)$ and $(-1, -1, 8)$ (after scaling by 28) respectively. The reflected normal is a vector (a, b, c) such that $(0, 0, 1), (a, b, c)$ have the same magnitude and their sum is a scalar multiple of $(-1, -1, 8)$. So $a = b$, and $\frac{c+1}{8} = \frac{a}{-1}$. Plugging this into $a^2 + b^2 + c^2 = 1$, we get $a^2 + a^2 + (8a + 1)^2 = 1$, which simplifies to $66a^2 + 16a = 0$. Discarding the $a = 0$ root (which gives us $(0, 0, -1)$, not the vector we want), we get $a = -\frac{8}{33}$. Then $c = \frac{31}{33}$, so the answer is $-\frac{31}{8}$.

15. Find the number of pairs of positive integers a, b, with $a \leq 125$ and $b \leq 100$, such that $a^b - 1$ is divisible by 125.

    Solution: The answer is 520. It is a well-known result in number theory that mod p^k has primitive roots for any odd prime p and positive integer k, and $125 = 5^3$. For $a^b \equiv 1 \pmod{125}$ we require a to be a unit mod 125, so therefore we can write it as g^c where g is an arbitrary primitive root and c is a positive integer at most 100. Then we need $g^{bc} \equiv 1 \pmod{125}$ and the question is how many pairs of positive integers b, c with $b, c \leq 100$ have bc divisible by 100. Let $f(n)$ be the number of ways to do this where 100 is replaced by n. By the Chinese Remainder Theorem we can determine that $f(4)f(25) = f(100)$, so it suffices to compute $f(p^2)$ for a prime p.

    There are two cases: either one b, c is equal to p^2, or neither is and both are divisible by p. In the first case, we have p^2 choices for c if $b = p^2$, and likewise p^2 choices for b if $c = p^2$. This overcounts the $b = c = p^2$ case once, for a total of $2p^2 - 1$. In the second case, we have $b = px$ and $c = py$ for $x, y \in \{1, 2, \ldots p-1\}$ for $(p-1)^2$ choices. Adding all of these up, we get $3p^2 - 2p$. So $f(4) = 8, f(25) = 65$, and $f(100) = 520$.