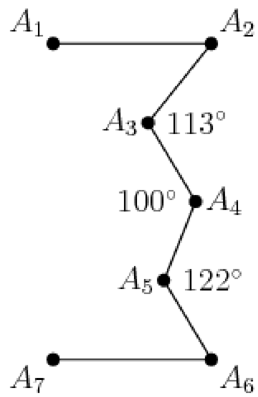
Caltech Harvey Mudd Mathematics Competition

Individual Round **March 3, 2012**

1. A robot is at position 0 on a number line. Each second, it randomly moves either one unit in the positive direction or one unit in the negative direction, with probability $\frac{1}{2}$ of doing each. Find the probability that after 4 seconds, the robot has returned to position 0.

2. How many positive integers $n \le 20$ are such that the greatest common divisor of n and 20 is a prime number?

3. A sequence of points $A_1, A_2, A_3, \ldots A_7$ is shown in the diagram below, with A_1A_2 parallel to A_6A_7. We have $\angle A_2A_3A_4 = 113°$, $\angle A_3A_4A_5 = 100°$, and $\angle A_4A_5A_6 = 122°$. Find the degree measure of $\angle A_1A_2A_3 + \angle A_5A_6A_7$.



4. Compute
$$\log_3\left(\frac{\log_3 3^{3^{3^3}}}{\log_{3^3} 3^{3^3}}\right).$$

5. In an 8×8 chessboard, a pawn has been placed on the third column and fourth row, and all the other squares are empty. It is possible to place nine rooks on this board such that no two rooks attack each other. How many ways can this be done? (Recall that a rook can attack any square in its row or column provided all the squares in between are empty.)

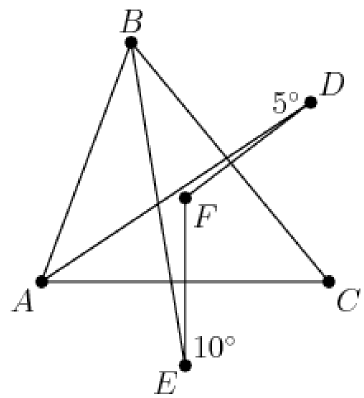
6. Suppose that a, b are positive real numbers with $a > b$ and $ab = 8$. Find the minimum value of $\frac{a^2+b^2}{a-b}$.

7. A cone of radius 4 and height 7 has A as its apex and B as the center of its base. A second cone of radius 3 and height 7 has B as its apex and A as the center of its base. What is the volume of the region contained in both cones?

8. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be a permutation of the numbers $1, 2, 3, 4, 5, 6$. We say a_i is *visible* if a_i is greater than any number that comes before it; that is, $a_j < a_i$ for all $j < i$. For example, the permutation $2, 4, 1, 3, 6, 5$ has three visible elements: $2, 4, 6$. How many such permutations have exactly two visible elements?

9. Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$, and let $S = [f(6)]^5 + [f(10)]^3 + [f(15)]^2$. Compute the remainder when S is divided by 30.

10. In triangle ABC, the angle bisector from A and the perpendicular bisector of BC meet at point D, the angle bisector from B and the perpendicular bisector of AC meet at point E, and the perpendicular bisectors of BC and AC meet at point F. Given that $\angle ADF = 5°$, $\angle BEF = 10°$, and $AC = 3$, find the length of DF.



11. Let $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. How many subsets S of $\{1, 2, \ldots 2011\}$ are there such that

$$F_{2012} - 1 = \sum_{i \in S} F_i?$$

12. Let a_k be the number of perfect squares m such that $k^3 \leq m < (k+1)^3$. For example, $a_2 = 3$ since three squares m satisfy $2^3 \leq m < 3^3$, namely $9, 16$, and 25. Compute

$$\sum_{k=0}^{99} \lfloor \sqrt{k} \rfloor a_k,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

13. Suppose that a, b, c, d, e, f are real numbers such that

$$a + b + c + d + e + f = 0,$$

$$a + 2b + 3c + 4d + 2e + 2f = 0,$$

$$a + 3b + 6c + 9d + 4e + 6f = 0,$$

$$a + 4b + 10c + 16d + 8e + 24f = 0,$$

$$a + 5b + 15c + 25d + 16e + 120f = 42,$$

Compute $a + 6b + 21c + 36d + 32e + 720f$.

14. In Cartesian space, three spheres centered at $(-2, 5, 4), (2, 1, 4)$, and $(4, 7, 5)$ are all tangent to the xy-plane. The xy-plane is one of two planes tangent to all three spheres; the second plane can be written as the equation $ax + by + cz = d$ for some real numbers a, b, c, d. Find $\frac{c}{a}$.

15. Find the number of pairs of positive integers a, b, with $a \leq 125$ and $b \leq 100$, such that $a^b - 1$ is divisible by 125.