2012 Spring CHMMC Power Round

In this round you will develop a proof of a number theoretic fact through mostly geometrical methods. For this part of the contest, you must fully justify all of your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems. Be sure to read the background information below before working on the problems.

On the coordinate plane, begin by drawing two circles of unit diameter which lie above the x-axis and are tangent to it at $(0,0)$ and $(1,0)$ respectively. Next, draw a smaller circle tangent to both of the original two circles and also tangent to the x-axis, as shown in Figure 1.

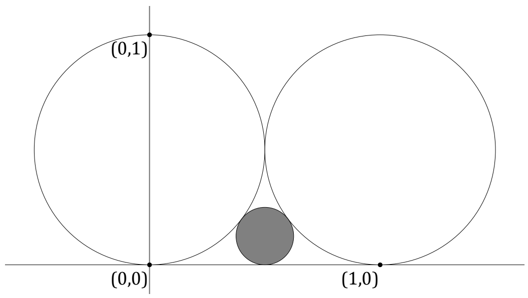


Figure 1

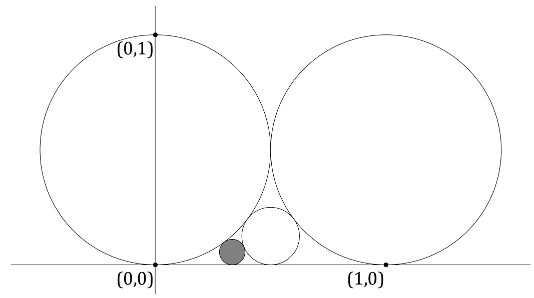


Figure 2

Select two circles that are tangent to each other, and as before draw a smaller circle between them tangent to both selected circles as well as the x-axis. Figure 2 above shows the next such circle that might be drawn. Continue this process indefinitely, selecting two tangent circles and drawing the smaller circle tangent to the x-axis and to both of them. After all of the circles have been drawn, of which there are infinitely many, you will get an image like in Figure 3.

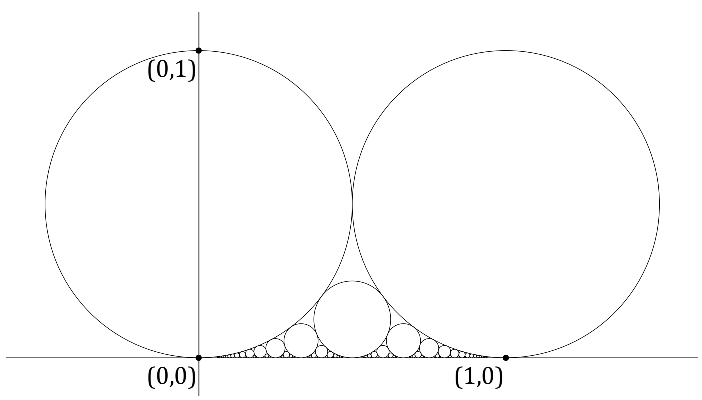


Figure 3

Throughout this round you should assume any fractions are in lowest terms. That is, whenever $\frac{x}{y}$ is written in a problem, x, y are assumed to be relatively prime.

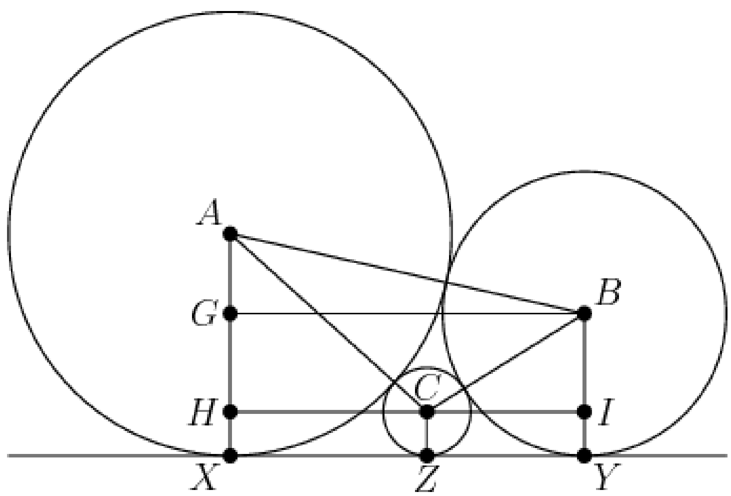
1. For each of the two shaded circles in Figures 1 and 2, find the diameter and point of tangency with the x-axis. For this problem only, you may give answers without any justification.

   Solution: The diameters of the shaded circles in Figures 1 and 2 are $\frac{1}{4}$ and $\frac{1}{9}$ respectively. Their points of tangency with the x-axis are $(\frac{1}{2}, 0)$ and $(\frac{1}{3}, 0)$ respectively.

2. Solve problem 1 in the general case: Suppose we have a circle C_1 with diameter a^2 tangent to the x-axis at X and a circle C_2 with diameter b^2 tangent to the x-axis at Y, with the two circles also tangent to each other. If we construct a smaller circle C_3 tangent to both C_1, C_2 as well as to the x-axis at Z, compute C_3's diameter and the ratio $\frac{XZ}{ZY}$. (Citing Descartes' circle formula in this problem will not get credit.)

   Solution: The diameter is $\frac{1}{\left(\frac{1}{a}+\frac{1}{b}\right)^2} = \frac{a^2b^2}{(a+b)^2}$ and the ratio is $\frac{a}{b}$.

   WLOG $a > b$. In the provided diagram, A, B, C are centers of circles C_1, C_2, C_3 respectively, and X, Y, Z are their points of tangency between a circle and the x-axis. We construct G on AX so that CG is perpendicular to AX, H on AX so that BH is perpendicular to AX, and similarly we construct I on CZ with BI perpendicular to CZ.



   Set $x = \frac{a^2}{2}, y = \frac{b^2}{2}$, and z equal to the radius of the circle we want. Note that $AX = x$, $BY = z$, $CZ = y$, $AB = x + z$, and $BC = y + z$. We also note that since AX, BY, CZ are parallel, we have that $GCZX$, $HBYX$, and $BIZY$ are rectangles, so $GX = CZ = y$, $HX = IZ = BY = z$. This gives us $AG = AX - GX = x - y$. By the Pythagorean Theorem to right triangle AGC, we get $GC^2 + (x - y)^2 = (x + y)^2$, which tells us $GC = 2\sqrt{xy}$. From this and $GCHI$ being a rectangle we have $HI = 2\sqrt{xy}$.

   We have $AH = AX - HX = x - z$ and $CI = CZ - IZ = y - z$. Let the length of $HB = w$, then $BI = HI - HB = 2\sqrt{xy} - w$ from the previous paragraph. Applying the Pythagorean Theorem to right triangles AHB, CIB, we get the following system of equations:

$$(x - z)^2 + w^2 = (x + z)^2,$$

$$(y - z)^2 + (2\sqrt{xy} - w)^2 = (y + z)^2.$$

The top one simplifies to $2\sqrt{xz} = w$. The bottom simplifies to $2\sqrt{yz} = 2\sqrt{xy} - w$. Therefore $\sqrt{yz}+\sqrt{xz} = \sqrt{xy}$. Factoring a $\sqrt{z}$ out from the left, isolating it, and squaring, we get $z = \frac{xy}{(\sqrt{x}+\sqrt{y})^2}$. Substituting $x = \frac{a^2}{2}, y = \frac{b^2}{2}$ back in, we get

$$2z = \frac{a^2b^2}{(a+b)^2}.$$

To get $\frac{XZ}{ZY}$, recall that we computed that $XZ = w = 2\sqrt{xz}$ and $YZ = 2\sqrt{xy} - w = 2\sqrt{yz}$. The ratio of these two is $\sqrt{\frac{x}{y}}$, which after reversing the substitution $x = \frac{a^2}{2}, y = \frac{b^2}{2}$ gives $\frac{a}{b}$ as the ratio.

3. Suppose that two circles tangent to the x-axis at $(\frac{a}{b}, 0)$ and $(\frac{c}{d}, 0)$ have diameters $\frac{1}{b^2}$ and $\frac{1}{d^2}$ respectively. Show that they are tangent if and only if $|ad - bc| = 1$. (Assume $\frac{a}{b}, \frac{c}{d}$ are written in lowest terms.)

   Solution: The centers of the two circles are $B = (\frac{a}{b}, \frac{1}{2b^2})$ and $D = (\frac{c}{d}, \frac{1}{2d^2})$ respectively. These circles are tangent if and only if the distance between these two points is equal to $\frac{1}{2b^2} + \frac{1}{2d^2}$. By the distance formula, we require that

$$\left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2 = \left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2,$$

   Moving the last term over and applying difference of squares,

$$\frac{1}{b^2d^2} = \left(\frac{a}{b} - \frac{c}{d}\right)^2,$$

$$1 = (ad - bc)^2.$$

   This is equivalent to $|ad - bc| = 1$, which shows both directions and completes the proof.

4. Let $\frac{p}{q}$ be a rational number in lowest terms with $0 < \frac{p}{q} < 1$. Show that there exists a pair of rational numbers $\frac{a}{b}, \frac{c}{d}$ (both written in lowest terms) such that $0 \leq \frac{a}{b}, \frac{c}{d} \leq 1$, $|aq - bp| = |cq - dp| = 1$, $a + c = p$, and $b + d = q$. (Hint: You are not expected to use geometry to solve this.)

   Solution: By Bezout's Lemma we can find integers x, y such that $px + qy = 1$. Let $a = kp + y$ and $b = kq - x$ with k picked so that $0 < a \leq p$. Then $aq - bp = 1$. Let $c = p - a$ and $d = q - b$. This implies immediately that $|cq - dp| = 1$, $a + c = p$, and $b + d = q$.

   We now need to show that $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers between 0 and 1 inclusive. It suffices to prove that $0 < a \leq b$ and $0 \leq c < d$. We defined a to satisfy $0 < a \leq p$, which in turn implies $0 \leq c < p$, so the two left inequalities are satisfied. From $p < q$, we have $\frac{q}{p} > 1$, so

$$b = \frac{qa - 1}{p} \geq \frac{q}{p}(a - 1) > a - 1 \Rightarrow b \geq a,$$

$$d = q - b = \frac{qp - (qa - 1)}{p} = \frac{qc + 1}{p} \geq \frac{q}{p}c > c \Rightarrow d > c.$$

   This completes the proof.

5. (a) For a rational $\frac{p}{q}$ written in lowest terms with $0 < \frac{p}{q} < 1$, show that there is a circle of Figure 3 tangent to the x-axis at $(\frac{p}{q}, 0)$ with a diameter of $\frac{1}{q^2}$.

   Solution: We show this by strong induction on the denominator q, extending the problem to $0 \leq \frac{p}{q} \leq 1$. The base case $q = 1$ corresponds to the two circles we started with. Now suppose we know the induction hypothesis is true up to q, and let $\frac{p}{q}$ be a rational number in lowest terms with $0 < p < q$.

Using the result of problem 4, we can find two rational numbers $\frac{a}{b}, \frac{c}{d}$ between 0 and 1 with $|aq - bp| = |cq - dp| = 1$ and $a + c = p, b + d = q$. In particular both denominators are strictly less than q. By the induction hypothesis there are circles tangent to $(\frac{a}{b}, 0)$ and $(\frac{c}{d}, 0)$ with respective diameters $\frac{1}{b^2}, \frac{1}{d^2}$. By problem 3 these circles are tangent, so a new circle was constructed between them at some point. Using the result of problem 2b, this circle is tangent to the x-axis at $(\frac{p}{q}, 0)$ and by 2a it has diameter $\frac{1}{q^2}$. This completes the induction and the proof.

(b) For r an irrational number satisfying $0 < r < 1$, show that no circle of Figure 3 is tangent to the x-axis at $(r, 0)$.

Solution: Suppose such a circle did exist tangent to the x-axis at $R = (r, 0)$, and take the first one found in the process of constructing the circles of Figure 3. Since it is the first such circle constructed, the two circles it comes from were tangent to the x-axis at $A = (a, 0)$, $B = (b, 0)$ for rational a, b. By problem 3a, these two circles have rational diameters, and so by problem 2b, the ratio $\frac{AR}{RB}$ is rational, so $r = a + \frac{AR}{AB}(b - a)$ is rational, contradiction.

6. Consider any of the bounded contiguous regions that are outside of all of the circles of the configuration. Figure 4 shows the largest of these regions shaded in.

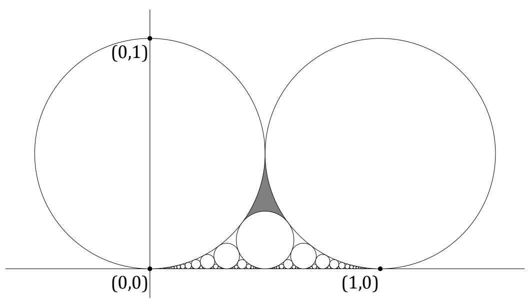


Figure 4

Show that any such region is surrounded by exactly three circles which are mutually externally tangent. (In particular none of these regions touch the x-axis.)

Solution: Let R be such a region. The boundary of R contains at least one point of tangency between two circles, because a single circle is not enough to make R bounded, and the only other possible points of tangency are on the x-axis and these are also not enough to make R bounded. Let this point of tangency be between circles C_1 and C_2, with C_2 the smaller one. If C_1, C_2 are the two original circles, then R is the region in Figure 4 and we are already done. Otherwise, C_1 and C_2 are both tangent to two other circles, one of them D such that C_2 was constructed to be between C_1 and D, and the other is D', the circle constructed between C_1 and C_2. Regardless of which side of the point of tangency R is on, R is a region bounded either by C_1, C_2, D or C_1, C_2, D', as desired.

7. A well-known theorem in elementary number theory is that for a real number α, the inequality

$$\left| \frac{p}{q} - \alpha \right| \leq \frac{1}{2q^2}$$

has infinitely many solutions in integers p, q with $q > 0$ if and only if α is irrational.

(a) Use the results of the previous problems to prove this theorem.

Solution: Consider the vertical line ℓ defined by $x = \alpha$. This line intersects the circle that is tangent to the x-axis at $(\frac{p}{q}, 0)$ if $\left|\frac{p}{q} - \alpha\right|$ is less than the radius of the circle. By 5a, such a circle exists and has radius $\frac{1}{q^2}$, so therefore ℓ intersects this circle if and only if $\frac{p}{q}$ is a solution to the inequality

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{2q^2}.$$

Suppose $\alpha = \frac{p}{q}$ is rational. Then as ℓ moves in the positive y-direction from the x-axis, it lies entirely inside the circle tangent to $(\frac{p}{q}, 0)$ until it exits at $(\frac{p}{q}, \frac{1}{q^2})$. From then on it can only intersect circles that have a larger diameter, of which there are finitely many.

Now suppose α is irrational. Start from $(\alpha, \frac{1}{2})$, which inside one of the two larger circles and move along ℓ in the negative y-direction. Suppose it enters and exits only a finite number of circles. Then when it reaches the x-axis at $(\alpha, 0)$, it either does so while inside a circle or while inside one of the regions defined by problem 6. If it was inside a circle, this circle is tangent to the x-axis at $(\alpha, 0)$, which contradicts the result of problem 5b. If it was inside a region outside of any circle, this region touches the x-axis, which contradicts the result of problem 6. So we have a contradiction in all cases, and ℓ intersects an infinite number of circles.

(b) For an irrational number α, let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots$ be the solutions to the equation of the previous problem with $q_1 < q_2 < q_3 < \ldots$. Show that $|p_k q_{k+1} - q_k p_{k+1}| = 1$.

Solution: As shown in the previous solution the line ℓ enters and exits bounded regions and circles an infinite number of times. Each circle it enters has a smaller diameter than the previous. As a result $\frac{p_i}{q_i}$ is the ith circle that ℓ enters, since $q_1 < q_2 < \cdots$ implies that $\frac{1}{q_1^2} > \frac{1}{q_2^2} > \cdots$ so that the diameters are in decreasing order.

Consider $\frac{p_k}{q_k}$ and $\frac{p_{k+1}}{q_{k+1}}$. The line ℓ exited the circle tangent to the x-axis at $(\frac{p_k}{q_k}, 0)$, travelled through a region outside the circles, and then entered the circle tangent to the x-axis at $(\frac{p_{k+1}}{q_{k+1}}, 0)$. These two circles are both part of the boundary for the same region, so by problem 6 they are tangent to each other. By problem 3, $|p_k q_{k+1} - q_k p_{k+1}| = 1$ as desired.