2012 Spring CHMMC Power Round

In this round you will develop a proof of a number theoretic fact through mostly geometrical methods. For this part of the contest, you must fully justify all of your answers unless otherwise specified. In your solutions, you may refer to the answers of earlier problems (but not later problems or later parts of the same problem), even if you were not able to solve those problems. Be sure to read the background information below before working on the problems.

On the coordinate plane, begin by drawing two circles of unit diameter which lie above the x-axis and are tangent to it at $(0, 0)$ and $(1, 0)$ respectively. Next, draw a smaller circle tangent to both of the original two circles and also tangent to the x-axis, as shown in Figure 1.

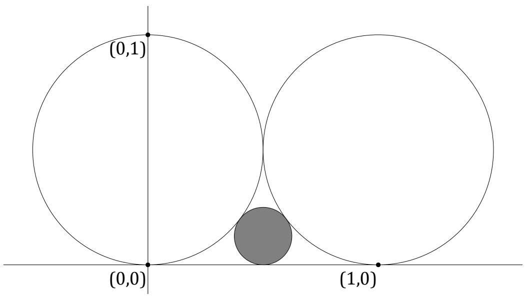

Figure 1

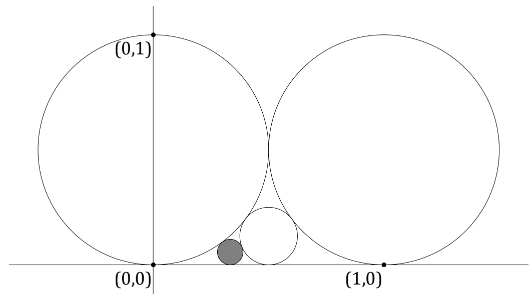

Figure 2

Select two circles that are tangent to each other, and as before draw a smaller circle between them tangent to both selected circles as well as the x-axis. Figure 2 above shows the next such circle that might be drawn. Continue this process indefinitely, selecting two tangent circles and drawing the smaller circle tangent to the x-axis and to both of them. After all of the circles have been drawn, of which there are infinitely many, you will get an image like in Figure 3.

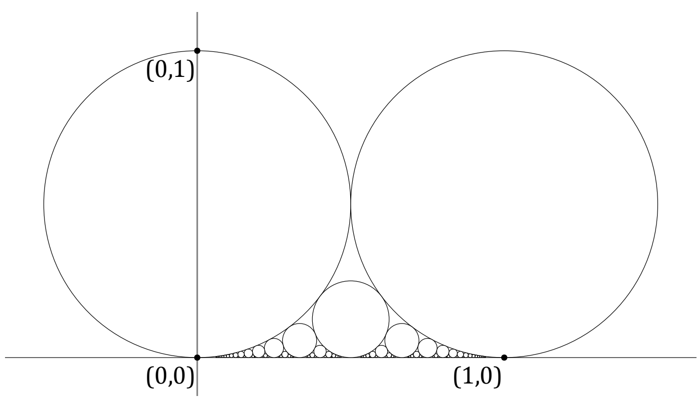

Figure 3

Throughout this round you should assume any fractions are in lowest terms. That is, whenever $\frac{x}{y}$ is written in a problem, x, y are assumed to be relatively prime.

1. For each of the two shaded circles in Figures 1 and 2, find the diameter and point of tangency with the x-axis. For this problem only, you may give answers without any justification.

2. Solve problem 1 in the general case: Suppose we have a circle C_1 with diameter a^2 tangent to the x-axis at X and a circle C_2 with diameter b^2 tangent to the x-axis at Y, with the two circles also tangent to each other. If we construct a smaller circle C_3 tangent to both C_1, C_2 as well as to the x-axis at Z, compute C_3's diameter and the ratio $\frac{XZ}{ZY}$. (Citing Descartes' circle formula in this problem will not get credit.)

3. Suppose that two circles tangent to the x-axis at $(\frac{a}{b}, 0)$ and $(\frac{c}{d}, 0)$ have diameters $\frac{1}{b^2}$ and $\frac{1}{d^2}$ respectively. Show that they are tangent if and only if $|ad - bc| = 1$. (Assume $\frac{a}{b}, \frac{c}{d}$ are written in lowest terms.)

4. Let $\frac{p}{q}$ be a rational number in lowest terms with $0 < \frac{p}{q} < 1$. Show that there exists a pair of rational numbers $\frac{a}{b}, \frac{c}{d}$ (both written in lowest terms) such that $0 \leq \frac{a}{b}, \frac{c}{d} \leq 1$, $|aq - bp| = |cq - dp| = 1$, $a + c = p$, and $b + d = q$. (Hint: You are not expected to use geometry to solve this.)

5. (a) For a rational $\frac{p}{q}$ written in lowest terms with $0 < \frac{p}{q} < 1$, show that there is a circle of Figure 3 tangent to the x-axis at $(\frac{p}{q}, 0)$ with a diameter of $\frac{1}{q^2}$.
   (b) For r an irrational number satisfying $0 < r < 1$, show that no circle of Figure 3 is tangent to the x-axis at $(r, 0)$.

6. Consider any of the bounded contiguous regions that are outside of all of the circles of the configuration. Figure 4 shows the largest of these regions shaded in.

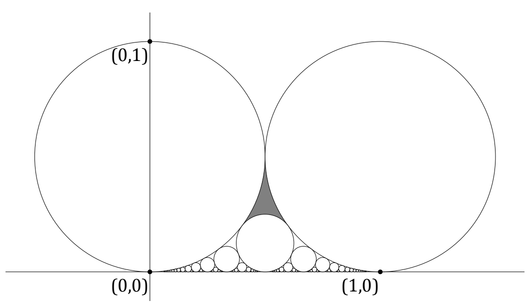

Figure 4

Show that any such region is surrounded by exactly three circles which are mutually externally tangent. (In particular none of these regions touch the x-axis.)

7. A well-known theorem in elementary number theory is that for a real number α, the inequality

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{2q^2}$$

has infinitely many solutions in integers p, q with $q > 0$ if and only if α is irrational.

   (a) Use the results of the previous problems to prove this theorem.
   (b) For an irrational number α, let $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots$ be the solutions to the equation of the previous problem with $q_1 < q_2 < q_3 < \ldots$. Show that $|p_k q_{k+1} - q_k p_{k+1}| = 1$.