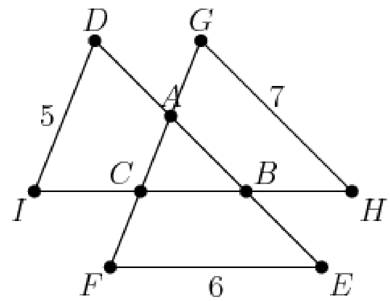
Caltech Harvey Mudd Mathematics Competition

1. Let a, b, c be positive integers. Suppose that $(a+b)(a+c) = 77$ and $(a+b)(b+c) = 56$. Find $(a+c)(b+c)$.

   Solution: The answer is 88. Note that $\gcd(77, 56) = 7$ and that $a+b$ divides $\gcd((a+b)(a+c), (a+b)(b+c)) = 7$. Since $a+b > 1$ and 7 is prime, we must have $a+b = 7$. Then $(a+c)(b+c) = \frac{77 \cdot 56}{7^2} = 11 \cdot 8 = 88$. (It's also possible to solve for a, b, c explicitly, getting $5, 2, 6$ respectively.)

2. In the diagram below, A and B trisect DE, C and A trisect FG, and B and C trisect HI. Given that $DI = 5$, $EF = 6$, $GH = 7$, find the area of $\triangle ABC$.



   Solution: The answer is $\frac{3\sqrt{6}}{2}$. Using equal sides and midpoints we can find $AB = GH/2 = 7/2$, $BC = EF/2 = 3$, and $CA = DI/2 = 5/2$. By Heron's formula, the area of triangle ABC is

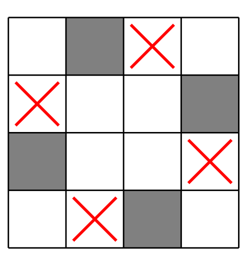
$$\sqrt{\frac{9}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{4}{2}} = \sqrt{\frac{27}{2}} = \frac{3\sqrt{6}}{2}.$$

3. In a 4×4 grid of sixteen unit squares, exactly 8 are shaded so that each shaded square shares an edge with exactly one other shaded square. How many ways can this be done?

   Solution: The answer is 15. Partition the grid into four 2×2 sections. Note that we cannot shade three squares in any of these sections, since then one shaded square would have to neighboring ones. So each section contains exactly two shaded squares.

   Suppose that in one of these sections the two squares do not touch. It is fairly quick to see that no matter where this occurs, this forces the configuration where the eight edge squares are shaded and the four corner and four interior squares are not. So there is one possibility in this case.

   Otherwise, in each section the two squares are adjacent. In the top left section, exactly one of row 2 column 1 or row 1 column 2 will be shaded. WLOG row 1 column 2 is; we will double the number of possibilities we get here to account for the other case. Since we know each pair of shaded squares is contained within a single section, we can make the deductions in the diagram below.



If none of the four interior squares are used, then all of the pairs are determined and there is one way. If one of the four interior squares is used, we can choose one of the four pairs that uses an interior square and the rest are determined for four ways. If two of the four interior squarse are used, then the interior squares have a checkerboard pattern for two ways.

So the answer is $1 + 2 \cdot (1 + 4 + 2) = 15$.

4. Let $P(x)$ be a monic polynomial of degree 3. Suppose that $P(x)$ has remainder $R(x)$ when it is divided by $(x-1)(x-4)$ and $2R(x)$ when it is divided by $(x-2)(x-3)$. Given that $P(0) = 5$, find $P(5)$.

   Solution: The answer is 15. Let $P(x) = x^3 + ax^2 + bx + 5$. We know that $R(x)$ has degree at most 1; let it be $R(x) = cx + d$. We also know that

$$P(x) = (x+p)(x-1)(x-4) + R(x),$$

$$P(x) = (x+q)(x-2)(x-3) + 2R(x).$$

   Using the x^2 terms, the first equation gives $a = p - 1 - 4$ and the second gives $a = q - 2 - 3$. Therefore $p = q$. Using the x terms, the first equation gives $b = -5p + 4 + c$ and the second gives $b = -5p + 6 + 2c$. Therefore $c = -2$. Using the constant terms, the first equation gives $5 = 4p + d$. Finally, plugging $x = 5$ into the first equation,

$$P(5) = (5+p)(5-1)(5-4) + 5c + d = 20 + 4p + 5(-2) + d = 20 - 10 + 5 = 15.$$

   An alternate solution, starting from the $p = q$ step above, is to notice that when $x^2 - 5x + 2$ is divided by $(x-1)(x-4)$, it leaves half the remainder that it does when divided by $(x-2)(x-3)$. Therefore $x^2 - 5x + 2$ divides $P(x)$. Using $P(0) = 5$, we determine that the linear factor is $x + \frac{5}{2}$, so $P(x) = (x^2 - 5x + 2)(x + \frac{5}{2})$ and $P(5) = 15$.

5. Suppose S is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. How many different possible values are there for the product of the elements in S?

   Solution: The answer is 52. First we count the number of products for $\{1, 2, 3, 4, 6\}$. There are four powers of 2 and two powers of 3 available, so the candidates are $2^a 3^b$ for $0 \le a \le 4$, $0 \le b \le 2$. The only two that cannot be constructed are 2^4 and 3^2, since these use all powers of 2 or all powers of 3 but are not divisible by 6. The other 13 candidates are easily constructed using a greedy algorithm.

   For each of the 13 solutions in that reduced problem, we can get four more in the original problem by multiplying it by $1, 5, 7$, or 35. Thus there are $4 \cdot 13 = 52$ solutions.

6. Compute

$$\prod_{k=1}^{12} \left(\prod_{j=1}^{10} \left(e^{2\pi ji/11} - e^{2\pi ki/13} \right) \right).$$

   (The notation $\prod_{k=a}^{b} f(k)$ means the product $f(a)f(a+1)\cdots f(b)$.)

   Solution: The answer is 1. Define the two polynomials

$$P(x) = 1 + x + x^2 + \cdots + x^{12} = \prod_{k=1}^{12} (x - e^{2\pi ki/13}),$$

$$Q(x) = 1 + x + x^2 + \cdots + x^{10} = \prod_{j=1}^{10}(x - e^{2\pi ji/11}).$$

So the given expression is equal to

$$\prod_{j=1}^{10} P(e^{2\pi ji/11}).$$

Note that $e^{2\pi ji/11}$ is a root of $Q(x)$ and that $P(x) = 1+x+x^2Q(x)$. So $P(e^{2\pi ji/11}) = 1+e^{2\pi ji/11}$ and our expression becomes

$$\prod_{j=1}^{10}(1 + e^{2\pi ji/11}) = \prod_{j=1}^{10}(-1 - e^{2\pi ji/11}) = Q(-1) = 1.$$

7. A positive integer x is *k-equivocal* if there exists two positive integers b, b' such that when x is represented in base b and base b', the two representations have digit sequences of length k that are permutations of each other. The smallest 2-equivocal number is 7, since 7 is 21 in base 3 and 12 in base 5. Find the smallest 3-equivocal number.

   Solution: The answer is 22. This is 211 in base 3 and 112 in base 4.

   Let n be the minimum; the example above shows $n \le 22$. It can be easily checked that a base 2 sequence with 3 digits can't be permuted to form the same number in another base. Since $n < 5^2 = 25$, the two bases must be 3 and 4. In particular, $n \ge 16$. It can be checked that $16, 17$ will fail, so the base 3 representation starts with a 2 and the base 4 representation starts with a 1. The two representations have to be permutations of one of $210, 211, 221$. It can be checked that any ordering of 210 fails, and for 211 the base 3 representation must have the 2 in front and be 211, which works. 212 will result in a higher n no matter how it is permuted. So 22 is the smallest possible.

8. A special kind of chess knight is in the origin of an infinite grid. It can make one of twelve different moves: it can move directly up, down, left, or right one unit square, or it can move 1 units in one direction and 3 units in an orthogonal direction. How many different squares can it be on after 2 moves?

   Solution: The answer is 65. To start, if it makes no knight-style moves there are 9 different squares it can end up on.

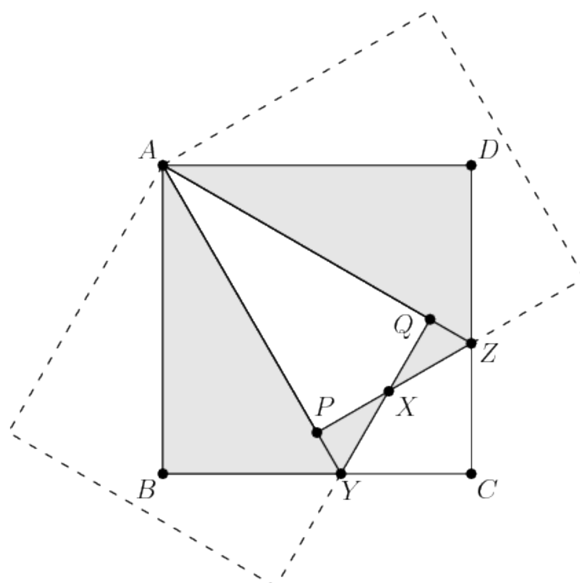
   If it makes one knight-style move and one single-unit move, note that moves commute so we can treat the knight-style move as being first. There are eight options for it and four options for the single-unit move. However, the squares $(0, 3), (3, 0), (-3, 0), (0, -3)$ can be reached in two different ways, so we get a total of $8 \cdot 4 - 4 = 28$ possibilities. Note that this is the only case where the sum of coordinates on the final square is odd, so there can be no overlap with other cases here.

   For the case of two different knight moves, note that if the knight moves 3 units of one direction and then 3 units in the reverse on its next move, it will land on a square already counted in the first case. So either it moves 3 units twice in the same direction, or 3 units in one direction and 3 units in an orthogonal one. Either way will avoid overlap with the first case. If it moves 3 units twice in the same direction, say positive y, it can end up on $(2, 6), (0, 6), (-2, 6)$. Account for all directions, we get 12 possibilities here. If it moves 3 units in one direction and 3 units in an orthogonal one, then WLOG it moves 3 units in the positive x and y directions (remember that moves commute so order does not matter). This gives us the four possibilities $(4,4),(4,2),(2,4),(2,2)$. There are four different quadrants it can land in, so this gives 16 possibilities here.

This accounts for all cases and gets a total of $9+28+12+16=65$.

9. Let S be a square of side length 1, one of whose vertices is A. Let S^+ be the square obtained by rotating S clockwise about A by 30°. Let S^- be the square obtained by rotating S counterclockwise about A by 30°. Compute the total area that is covered by exactly two of the squares S, S^+, S^-. Express your answer in the form $a+b\sqrt{3}$ where a,b are rational numbers.

   Solution: The answer is $-4+\frac{8}{3}\sqrt{3}$. Let the vertices of the square be A,B,C,D in anti-clockwise order. Let P be the image of B under the anti-clockwise rotation and Q the image of D under the clockwise rotation. Let lines AP and BC itnersect at Y, and similarly let lines AQ and CD intersect at Z. Since AQ bisects $\angle DAP$, S^- and S are symmetric about the line AZ, so therefore $\triangle ADZ$ is the reflection of $\triangle APZ$ about line AZ and thus the two triangles are congruent. Similarly, AP bisects $\angle QAB$, so $\triangle ABY \cong \triangle AQY$.



   If X is the intersection of QY and PZ, then X lies on the diagonal AC of S, so in particular $\triangle APX \cong AQX$ since they are reflections of each other across AX. The area we want is

$$K=[APZD]+[AQYB]-2[APXQ]=4[ABY]-4[APX].$$

   As ABY and APX are right triangles, the measure of $\angle A$ in each is 30°, 15° respectively, and $AB=AP=3$, their respective areas are $\frac{1}{2}\tan(30°)$ and $\frac{1}{2}\tan(15°)$. These simplify to $\frac{\sqrt{3}}{3}$ and $\frac{2-\sqrt{3}}{2}$. Taking their difference and multiplying by four gives us $\frac{8}{3}\sqrt{3}-4$, the answer.

   Thanks to aopsvd of the AOPS forums for sending this solution, which was better than the author's original.

10. A convex polygon in the Cartesian plane has all of its vertices on integer coordinates. One of the sides of the polygon is AB where $A=(0,0)$ and $B=(51,51)$, and the interior angles at A and B are both at most 45 degrees. Assuming no 180 degree angles, what is the maximum number of vertices this polygon can have?

Solution: The answer is 21.

Let $v_1, v_2, \ldots v_k$ be a sequence of 2D vectors with nonnegative integer entries, no two vectors collinear, and such that the entries of $v_1 + v_2 + \cdots + v_k$ are both at most 51. We now show $k \leq 20$. The first 19 possible vectors with the smallest sum of entries and no shorter vector in the same direction are, in order,

$$(1,0),(0,1),(1,1),(2,1),(1,2),(3,1),(1,3),(4,1),(3,2),$$

$$(2,3),(1,4),(5,1),(1,5),(6,1),(5,2),(4,3),(3,4),(2,5),(1,6),\ldots$$

This is 19 vectors with total sum $(46, 46)$. (A slightly more methodical way without listing all these out is to use the Euler phi function ϕ. One can compute that $1+\sum_{i=2}^{7} \frac{i\phi(i)}{2} = 46$, with the sum being greater than 50 if another term of the sum were included, and then $2 + \sum_{i=2}^{7} \phi(i) = 19$. The terms on the outside come from the $i = 1$ term being a little special for both sums.)

Any other vector has sum of entries at least 8, so any set of 21 vectors has total sum of components at least $46 + 46 + 2 \cdot 8 = 108$, which is attained by using the 19 vectors above and two vectors with sum of components 8. Since this is greater than $2 \cdot 51 = 102$, it is impossible to have 21 vectors in the sequence.

Notice that if we follow the sides of the polygon from A to B along each of the other vertices, we will construct a sequence of vectors satisfying all the conditions above. Hence there can be at most 20 such sides that we follow. Counting AB, this means the polygon can have at most 21 sides.

It remains to show that such a 21-gon can be constructed. Take the list of 19 vectors above along with $(3, 5)$, and order them by the angle they make with the vector $(1, 0)$ from smallest to largest (so the list starts with $(1, 0)$ and ends with $(0, 1)$). Make the sides of the polygon correspond to the vectors in this order and then replace $(1, 0)$ with $(3, 0)$. The increasing angle condition makes the polygon convex, the interior angles at A and B are exactly 45 degrees, and the sum of all the vectors is $(51, 51)$.