1. Let \(a, b, c\) be positive integers. Suppose that \((a + b)(a + c) = 77\) and \((a + b)(b + c) = 56\). Find \((a + c)(b + c)\).

2. In the diagram below, \(A\) and \(B\) trisect \(DE\), \(C\) and \(A\) trisect \(FG\), and \(B\) and \(C\) trisect \(HI\). Given that \(DI = 5\), \(EF = 6\), \(GH = 7\), find the area of \(\triangle ABC\).

3. In a \(4 \times 4\) grid of sixteen unit squares, exactly 8 are shaded so that each shaded square shares an edge with exactly one other shaded square. How many ways can this be done?

4. Let \(P(x)\) be a monic polynomial of degree 3. Suppose that \(P(x)\) has remainder \(R(x)\) when it is divided by \((x - 1)(x - 4)\) and \(2R(x)\) when it is divided by \((x - 2)(x - 3)\). Given that \(P(0) = 5\), find \(P(5)\).

5. Suppose \(S\) is a subset of \(\{1, 2, 3, 4, 5, 6, 7\}\). How many different possible values are there for the product of the elements in \(S\)?

6. Compute
\[
\prod_{k=1}^{12} \left( \prod_{j=1}^{10} \left( e^{2\pi ji/11} - e^{2\pi ki/13} \right) \right).
\]
(The notation \(\prod_{k=a}^{b} f(k)\) means the product \(f(a)f(a + 1) \cdots f(b)\).)

7. A positive integer \(x\) is \(k\)-equivocal if there exists two positive integers \(b, b'\) such that when \(x\) is represented in base \(b\) and base \(b'\), the two representations have digit sequences of length \(k\) that are permutations of each other. The smallest 2-equivocal number is 7, since 7 is 21 in base 3 and 12 in base 5. Find the smallest 3-equivocal number.

8. A special kind of chess knight is in the origin of an infinite grid. It can make one of twelve different moves: it can move directly up, down, left, or right one unit square, or it can move 1 units in one direction and 3 units in an orthogonal direction. How many different squares can it be on after 2 moves?

9. Let \(S\) be a square of side length 1, one of whose vertices is \(A\). Let \(S^+\) be the square obtained by rotating \(S\) clockwise about \(A\) by 30°. Let \(S^-\) be the square obtained by rotating \(S\) counterclockwise about \(A\) by 30°. Compute the total area that is covered by exactly two of the squares \(S, S^+, S^-.\) Express your answer in the form \(a + b\sqrt{3}\) where \(a, b\) are rational numbers.

10. A convex polygon in the Cartesian plane has all of its vertices on integer coordinates. One of the sides of the polygon is \(AB\) where \(A = (0, 0)\) and \(B = (51, 51)\), and the interior angles at \(A\) and \(B\) are both at most 45 degrees. Assuming no 180 degree angles, what is the maximum number of vertices this polygon can have?