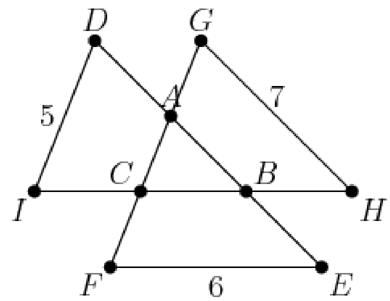
Caltech Harvey Mudd Mathematics Competition

Team Round **March 3, 2012**

1. Let a, b, c be positive integers. Suppose that $(a+b)(a+c) = 77$ and $(a+b)(b+c) = 56$. Find $(a+c)(b+c)$.

2. In the diagram below, A and B trisect DE, C and A trisect FG, and B and C trisect HI. Given that $DI = 5$, $EF = 6$, $GH = 7$, find the area of $\triangle ABC$.



3. In a 4×4 grid of sixteen unit squares, exactly 8 are shaded so that each shaded square shares an edge with exactly one other shaded square. How many ways can this be done?

4. Let $P(x)$ be a monic polynomial of degree 3. Suppose that $P(x)$ has remainder $R(x)$ when it is divided by $(x-1)(x-4)$ and $2R(x)$ when it is divided by $(x-2)(x-3)$. Given that $P(0) = 5$, find $P(5)$.

5. Suppose S is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. How many different possible values are there for the product of the elements in S?

6. Compute
$$\prod_{k=1}^{12} \left(\prod_{j=1}^{10} \left(e^{2\pi ji/11} - e^{2\pi ki/13} \right) \right).$$
(The notation $\prod_{k=a}^{b} f(k)$ means the product $f(a)f(a+1)\cdots f(b)$.)

7. A positive integer x is k-*equivocal* if there exists two positive integers b, b' such that when x is represented in base b and base b', the two representations have digit sequences of length k that are permutations of each other. The smallest 2-equivocal number is 7, since 7 is 21 in base 3 and 12 in base 5. Find the smallest 3-equivocal number.

8. A special kind of chess knight is in the origin of an infinite grid. It can make one of twelve different moves: it can move directly up, down, left, or right one unit square, or it can move 1 units in one direction and 3 units in an orthogonal direction. How many different squares can it be on after 2 moves?

9. Let S be a square of side length 1, one of whose vertices is A. Let S^+ be the square obtained by rotating S clockwise about A by $30°$. Let S^- be the square obtained by rotating S counterclockwise about A by $30°$. Compute the total area that is covered by exactly two of the squares S, S^+, S^-. Express your answer in the form $a + b\sqrt{3}$ where a, b are rational numbers.

10. A convex polygon in the Cartesian plane has all of its vertices on integer coordinates. One of the sides of the polygon is AB where $A = (0, 0)$ and $B = (51, 51)$, and the interior angles at A and B are both at most 45 degrees. Assuming no 180 degree angles, what is the maximum number of vertices this polygon can have?