Caltech Harvey Mudd Mathematics Competition

Part 1

1. Two kids A and B play a game as follows: From a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that:

   1. A goes first.
   2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and n, inclusive.
   3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive.

   The winner is the one who takes the last marble. What is the sum of all n for which B has a winning strategy?

2. How many ways can your rearrange the letters of "Alejandro" such that it contains exactly one pair of adjacent vowels?

3. Assuming real values for p, q, r, and s, the equation

   $$x^4 + px^3 + qx^2 + rx + s$$

   has four non-real roots. The sum of two of these roots is $q + 6i$, and the product of the other two roots is $3 - 4i$. Find the smallest value of q.

4. Lisa has a 3D box that is 48 units long, 140 units high, and 126 units wide. She shines a laser beam into the box through one of the corners, at a $45°$ angle with respect to all of the sides of the box. Whenever the laser beam hits a side of the box, it is reflected perfectly, again at a $45°$ angle. Compute the distance the laser beam travels until it hits one of the eight corners of the box.

Part 2

5. How many ways can you divide a heptagon into five non-overlapping triangles such that the vertices of the triangles are vertices of the heptagon?

6. Let a be the greatest root of $y = x^3 + 7x^2 - 14x - 48$. Let b be the number of ways to pick a group of a people out of a collection of a^2 people. Find $\frac{b}{2}$.

7. Consider the equation

   $$1 - \frac{1}{d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

   with a, b, c,and d being positive integers. What is the largest value for d?

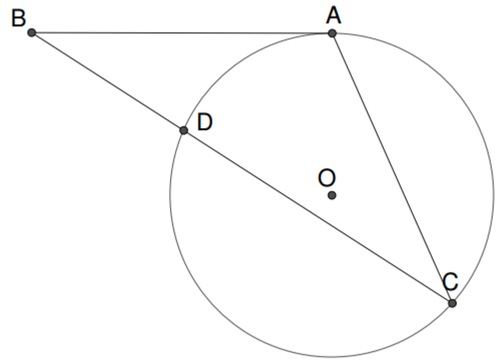
8. The number of non-negative integers $x_1, x_2, \ldots x_{12}$ such that

   $$x_1 + x_2 + \ldots + x_{12} \leq 17$$

   can be expressed in the form $\binom{a}{b}$, where $2b \leq a$. Find $a + b$.

Part 3

9. In the diagram below, AB is tangent to circle O. Given that $AC = 15$, $AB = 27/2$, and $BD = 243/34$, compute the area of $\triangle ABC$.



10. If

$$[2^{\log x}]^{[x^{\log 2}]^{[2^{\log x}]^{\cdots}}} = 2,$$

where $\log x$ is the base-10 logarithm of x, then it follows that $x = \sqrt{n}$. Compute n^2.

11.

12. Find n in the equation

$$133^5 + 110^5 + 84^5 + 27^5 = n^5,$$

where n is an integer less than 170.

Part 4

13. Let x be the answer to number 14, and z be the answer to number 16. Define $f(n)$ as the number of distinct two-digit integers that can be formed from digits in n. For example, $f(15) = 4$ because the integers $11, 15, 51, 55$ can be formed from digits of 15.

    Let w be such that $f(3xz - w) = w$. Find w.

14. Let w be the answer to number 13 and z be the answer to number 16. Let x be such that the coefficient of $a^x b^x$ in $(a+b)^{2x}$ is $5z^2 + 2w - 1$. Find x.

15. Let w be the answer to number 13, x be the answer to number 14, and z be the answer to number 16. Let A, B, C, D be points on a circle, in that order, such that $\overline{AD}$ is a diameter of the circle. Let E be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, let F be the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and let G be the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{AD}$. Now, let $AE = 3x$, $ED = w^2 - w + 1$, and $AD = 2z$. If $FG = y$, find y.

16. Let w be the answer to number 13, and x be the answer to number 16. Let z be the number of integers n in the set $S = \{w, w+1 \ldots 16x - 1, 16x\}$ such that $n^2 + n^3$ is a perfect square. Find z.