Individual Round

2014 CHMMC

Problem 1. *In the following 3 by 3 grid, a, b, c are numbers such that the sum of each row is listed at the right and the sum of each column is written below it:*

a	a	a	33
b	b	c	55
b	c	c	50
51	n	41	

What is n?

Problem 2. *Suppose in your sock drawer of 14 socks there are 5 different colors and 3 different lengths present. One day, you decide you want to wear two socks that have* both *different colors* and *different lengths. Given only this information, what is the maximum number of choices you might have?*

Problem 3. *The population of Arveymuddica is 2014, which is divided into some number of equal groups. During an election, each person votes for one of two candidates, and the person who was voted for by 2/3 or more of the group wins. When neither candidate gets 2/3 of the vote, no one wins the group. The person who wins the most groups wins the election. What should the size of the groups be if we want to minimize the minimum total number of votes required to win an election?*

Problem 4. *A farmer learns that he will die at the end of the year (day 365, where today is day 0) and that he has a number of sheep. He decides that his utility is given by ab where a is the money he makes by selling his sheep (which always have a fixed price) and b is the number of days he has left to enjoy the profit; i.e., $365 - k$ where k is the day. If every day his sheep breed and multiply their numbers by 103/101 (yes, there are small, fractional sheep), on which day should he sell them all?*

Problem 5. *Line segements $\overline{AB}$ and $\overline{AC}$ are tangent to a convex arc $\widehat{BC}$ and $\angle BAC = \frac{\pi}{3}$. If $\overline{AB} = \overline{AC} = 3\sqrt{3}$, find the length of $\widehat{BC}$.*

Problem 6. *Suppose that you start with the number 8 and always have two legal moves:*

- *Square the number*
- *Add one if the number is divisible by 8 or multiply by 4 otherwise*

How many sequences of 4 moves are there that return to a multiple of 8?

Problem 7. *A robot is shuffling a 9 card deck. Being very well machined, it does every shuffle in exactly the same way: it splits the deck into two piles, one containing the 5 cards from the bottom of the deck and the other with the 4 cards from the top. It then interleaves the cards from the two piles, starting with a card from the bottom of the larger pile at the bottom of the new deck, and then alternating cards from the two piles while maintaining the relative order of each pile. The top card of the new deck will be the top card of the bottom pile.*

The robot repeats this shuffling procedure a total of n times, and notices that the cards are in the same order as they were when it started shuffling. What is the smallest possible value of n?

Problem 8. *A secant line incident to a circle at points A and C intersects the circle's diameter at point B with a 45° angle. If the length of AB is 1 and the length of BC is 7, then what is the circle's radius?*

Problem 9. *If a complex number z satisfies $z + 1/z = 1$, then what is $z^{96} + 1/z^{96}$?*

Problem 10. *Let a, b be two acute angles where $\tan a = 5 \tan b$. Find the maximum possible value of $\sin(a - b)$.*

Problem 11. *A pyramid, represented by $SABCD$ has parallelogram $ABCD$ as base (A is across from C) and vertex S. Let the midpoint of edge SC be P. Consider plane $AMPN$ where M is on edge SB and N is on edge SD. Find the minimum value r_1 and maximum value r_2 of*

$$\frac{V_1}{V_2}$$

where V_1 is the volume of pyramid $SAMPN$ and V_2 is the volume of pyramid $SABCD$. Express your answer as an ordered pair (r_1, r_2).

Problem 12. *A 5x5 grid is missing one of its main diagonals. In how many ways can we place 5 pieces on the grid such that no two pieces share a row or column?*

Problem 13. *There are 20 cities in a country, some of which have highways connecting them. Each highway goes from one city to another, both ways. There is no way to start in a city, drive along the highways of the country such that you travel through each city exactly once, and return to the same city you started in. What is the maximum number of roads this country could have?*

Problem 14. *Find the area of the cyclic quadrilateral with side lengths given by the solutions to $x^4 - 10x^3 + 34x^2 - 45x + 19 = 0$.*

Problem 15. *Suppose that we know $u_{0,m} = m^2 + m$ and $u_{1,m} = m^2 + 3m$ for all integers m, and that*

$$u_{n-1,m} + u_{n+1,m} = u_{n,m-1} + u_{n,m+1}$$

Find $u_{30,-5}$.