Power Round

Caltech-Harvey Mudd Math Competition

Fall 2014

In this problem, we will derive various properties of Dirichlet Convolutions, a powerful tool in number theory. Consider two real functions f and g whose domain is the positive integers. Then their convolution is a new function:

$$(f * g)(n) = \sum_{k|n} f(k)g(n/k)$$

In other words, the convolution of f and g at n is the sum of $f(k)g(n/k)$ over all positive divisors k of n.

1. To begin to understand the Dirichlet convolution, let

$$A(n) = \begin{cases} 1 & n = 2^k \text{ for some integer k} \\ 0 & \text{otherwise} \end{cases}$$

$$B(n) = n$$

and find a simple formula for the convolution $(A * B)(n)$ in terms of n's prime factorization $2^{k_1}3^{k_2}5^{k_3}\ldots$. (10 pts)

2. a. Prove that Dirichlet convolutions are
   - commutative ($f * g = g * f$), and
   - associative ($f * (g * h) = (f * g) * h$).

   (10 pts)

   b. Show that the function

$$\epsilon(n) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

   is the identity; that $(\epsilon * f)(n) = f(n)$, and that no other function has this property. (5 pts)

   c. A function f^{-1} is an *inverse* to a function f if $(f * f^{-1})(n) = \epsilon(n)$. Give
   - An argument that inverses exist by describing a process for computing the inverse of a given function. A full inductive definition of the inverse is not required.
   - A proof that the inverse is unique.

   (10 pts)

   Note that these first parts have proven that the set of these real-valued functions f on the positive integers with $f(1) \neq 0$ are a commutative *group* under Dirichlet inversion. Let this set of functions be U.

3. Next we would like try inverting the function $\mathbf{1}(n) = 1$. Its inverse is called the Möbius function μ; that is, the function μ such that $\sum_{k|n} \mu(k)\mathbf{1}(n/k) = \epsilon(n)$.

   a. Find $\mu(n)$ in the special cases that
   - $n = 1$.
   - $n = p$ is prime.
   - $n = p^2$ is the square of a prime.
   - $n = p_1p_2$ is the product of 2 primes.

   (8 pts)

   b. Use induction to find $\mu(p_1p_2\ldots p_\ell)$ where $p_1p_2\ldots p_\ell$ is the product of ℓ distinct prime factors. Hint: try to see a pattern in your results for 1, p, and p_1p_2 in the previous part. (8 pts)

c. Determine $\mu(n)$ where n's prime factorization contains at least one repeated prime factor, then summarize your results from this and the previous part in one expression (with multiple cases) for μ. Hint: divide the factors of n into two cases: those that have repeated prime factors and those that do not. (8 pts)

d. There exists a function f such that $\sum_{k|n} f(k) = n^2$ for all positive integers n. Using what you know about Dirichlet convolutions and μ, find $f(2^4 3^4)$. Your answer can be a prime factorization. (6 pts)

4. Next, we would like to understand the structure of the elements of U. Consider the following subsets of U:

- U_s (s for scalar), whose elements satisfy

$$f(n) = \begin{cases} r & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

  for some constant r.

- U_m (m for multiplicative), whose elements satisfy $f(1) = 1$ and $f(mn) = f(m)f(n)$ for any relatively prime numbers m, n.
- U_a (a for anti-multiplicative), whose elements satisfy $f(1) = 1$ and $f(p^k) = 0$ for any prime p and integer $k \geq 1$

a. Show that these three sets of functions are pairwise disjoint (no two have common elements) except for the identity function ϵ. (2 pts)

b. Show that any function f in U with $f(1) = 1$ can be expressed as the convolution of a function in U_m and a function in U_a. Hint: Using f, construct a multiplicative function g such that $h = g^{-1} * f$ is anti-multiplicative. (10 pts)

c. Using the previous result, show that any function f in U can be expressed as $g_s * g_m * g_a$ where g_s is in U_s, g_m is in U_m, and g_a is in U_a. (3 pts)

d. Suppose the prime factorization of n is $p_1^{k_1} p_2^{k_2} \dots p_\ell^{k_\ell}$ where each $k_i \geq 1$. Also let the exponent for 2 in n's factorization be k. Define the function $F(n)$ such that

$$F(n) = \begin{cases} \ell(\ell-1) + 2 & k \geq 2 \\ (\ell-1)^2 + 2 & k = 1 \\ \frac{\ell(\ell-1)}{2} + 1 & k = 0 \end{cases}$$

Determine a multiplicative function G in U_m and anti-multiplicative function H in U_a such that $G * H = F$. (10 pts)