In this problem, we will derive various properties of Dirichlet Convolutions, a powerful tool in number theory. Consider two real functions \( f \) and \( g \) whose domain is the positive integers. Then their convolution is a new function:

\[
(f * g)(n) = \sum_{k \mid n} f(k)g(n/k)
\]

In other words, the convolution of \( f \) and \( g \) at \( n \) is the sum of \( f(k)g(n/k) \) over all positive divisors \( k \) of \( n \).

1. To begin to understand the Dirichlet convolution, let

\[
A(n) = \begin{cases} 
1 & n = 2^k \text{ for some integer } k \\
0 & \text{otherwise}
\end{cases}
\]

\[B(n) = n\]

and find a simple formula for the convolution \((A * B)(n)\) in terms of \( n \)’s prime factorization \(2^k \cdot 3^{k_2} \cdot 5^{k_3} \cdots\). (10 pts)

2. a. Prove that Dirichlet convolutions are
   - commutative \((f * g = g * f)\), and
   - associative \((f * (g * h) = (f * g) * h)\).
   (10 pts)

b. Show that the function

\[
\epsilon(n) = \begin{cases} 
1 & n = 1 \\
0 & n \neq 1
\end{cases}
\]

is the identity; that \((\epsilon * f)(n) = f(n)\), and that no other function has this property. (5 pts)

c. A function \( f^{-1} \) is an inverse to a function \( f \) if \((f * f^{-1})(n) = \epsilon(n)\). Give

- An argument that inverses exist by describing a process for computing the inverse of a given function. A full inductive definition of the inverse is not required.
- A proof that the inverse is unique.
   (10 pts)

Note that these first parts have proven that the set of these real-valued functions \( f \) on the positive integers with \( f(1) \neq 0 \) are a commutative group under Dirichlet inversion. Let this set of functions be \( U \).

3. Next we would like try inverting the function \(1(n) = 1\). Its inverse is called the Möbius function \( \mu \); that is, the function \( \mu \) such that \( \sum_{k \mid n} \mu(k)1(n/k) = \epsilon(n) \).

a. Find \( \mu(n) \) in the special cases that

- \( n = 1 \).
- \( n = p \) is prime.
- \( n = p^2 \) is the square of a prime.
- \( n = p_1p_2 \) is the product of 2 primes.
   (8 pts)

b. Use induction to find \( \mu(p_1p_2\ldots p_k) \) where \( p_1p_2\ldots p_k \) is the product of \( \ell \) distinct prime factors. Hint: try to see a pattern in your results for 1, \( p \), and \( p_1p_2 \) in the previous part. (8 pts)
c. Determine $\mu(n)$ where $n$’s prime factorization contains at least one repeated prime factor, then summarize your results from this and the previous part in one expression (with multiple cases) for $\mu$. Hint: divide the factors of $n$ into two cases: those that have repeated prime factors and those that do not. (8 pts)

d. There exists a function $f$ such that $\sum_{k|n} f(k) = n^2$ for all positive integers $n$. Using what you know about Dirichlet convolutions and $\mu$, find $f(2^43^4)$. Your answer can be a prime factorization. (6 pts)

4. Next, we would like to understand the structure of the elements of $U$. Consider the following subsets of $U$:

- $U_s$ (s for scalar), whose elements satisfy

$$f(n) = \begin{cases} r & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

for some constant $r$.

- $U_m$ (m for multiplicative), whose elements satisfy $f(1) = 1$ and $f(mn) = f(m)f(n)$ for any relatively prime numbers $m, n$.

- $U_a$ (a for anti-multiplicative), whose elements satisfy $f(1) = 1$ and $f(p^k) = 0$ for any prime $p$ and integer $k \geq 1$

a. Show that these three sets of functions are pairwise disjoint (no two have common elements) except for the identity function $\epsilon$. (2 pts)

b. Show that any function $f$ in $U$ with $f(1) = 1$ can be expressed as the convolution of a function in $U_m$ and a function in $U_a$. Hint: Using $f$, construct a multiplicative function $g$ such that $h = g^{-1} * f$ is anti-multiplicative. (10 pts)

c. Using the previous result, show that any function $f$ in $U$ can be expressed as $g_s * g_m * g_a$ where $g_s$ is in $U_s$, $g_m$ is in $U_m$, and $g_a$ is in $U_a$. (3 pts)

d. Suppose the prime factorization of $n$ is $p_1^{k_1}p_2^{k_2} \ldots p_i^{k_i}$ where each $k_i \geq 1$. Also let the exponent for 2 in $n$’s factorization be $k$. Define the function $F(n)$ such that

$$F(n) = \begin{cases} \ell(\ell - 1) + 2 & k \geq 2 \\ (\ell - 1)^2 + 2 & k = 1 \\ \ell(\ell - 1) + 1 & k = 0 \end{cases}$$

Determine a multiplicative function $G$ in $U_m$ and anti-multiplicative function $H$ in $U_a$ such that $G * H = F$. (10 pts)