Power Round Solutions

Caltech-Harvey Mudd Math Competition

Fall 2014

1. We have that the convolution is $n + n/2 + n/4 + \ldots n/2^{k_1}$, or $(2^{k_1} - 1)3^{k_2}5^{k_3}\ldots$.

2. a. To derive commutativity, substitute $d = n/k, k = n/d$. To derive associativity, notice

$$(f * (g * h))(n) = \sum_{k|n} \left(f(k) \sum_{d|(n/k)} g(d)h(n/kd) \right)$$

$$(f * (g * h))(n) = \sum_{\{k_1,k_2,k_3\}:n=k_1k_2k_3} f(k_1)g(k_2)h(k_3)$$

which does not depend on order we computed the convolution in.

b. Using the sum, we see that all terms drop except the $\epsilon(1)f(n)$ term, leaving $f(n)$. To see that no other function has this property, suppose for the sake of contradiction that g is another identity. Then for some n, $(g * f)(n)$ includes a nonzero term proportional to $f(k \neq n)$. Since $f(k)$ can be whatever we like, this will not be $f(n)$ in general.

c. To see that an inverse exists, we notice that expanding and rewriting $(g * g^{-1})(1) = 1$ gives $g^{-1}(1) = 1/g(1)$. Rewriting $(g * g^{-1})(p) = p$ for any prime gives $g^{-1}(p) = -g(p)/g(1)^2$ at that prime. Similarly, evaluating $g * g^{-1}$ at any product of primes and then rewriting gives the value of g^{-1} of that product, and so forth. In general, if the sum of the exponents in n's factorization is m, we can express $g^{-1}(n)$ in terms of terms depending only on $g^{-1}(n')$ and $g(n')$ where each n''s factorization has a sum of exponents of at most $m-1$. Thus we can inductively (or recursively) determine g^{-1}.

To see that it is unique, suppose f has two inverses g_1 and g_2. Then we have that $f * g_1 * g_2 = g_2$, but by associativity and commutativity it is also $f * g_2 * g_1 = g_1$. Therefore g_1 and g_2 are the same, so the inverse is unique.

3. a. Simple computation gives $\mu(1) = 1, \mu(p) = -1, \mu(p^2) = 0, \mu(p_1p_2) = 1$.

b. The correct formula is $\mu(p_1p_2\ldots p_\ell) = (-1)^\ell$. The base cases $\mu(1) = 1, \mu(p) = -1$ are already proven. The factors of $p_1p_2\ldots p_{\ell+1}$ can be separated into two types: those that have $p_{\ell+1}$ as a factor, and those that don't. Therefore,

$$0 = \sum_{k|p_1\ldots p_\ell} \mu(k) + \sum_{k|p_1\ldots p_\ell} \mu(kp_{\ell+1})$$

We know by the definition of μ that the first sum is 0. Using the binomial theorem and the inductive hypothesis, we get

$$0 = \sum_{r=1}^{\ell+1} \binom{\ell-1}{r-1}(-1)^r + \mu(p_1\ldots p_{\ell+1}) - (-1)^{\ell+1}$$

$$0 = -\sum_{r=0}^{\ell} \binom{\ell}{r}(-1)^r + \mu(p_1,\ldots p_{\ell+1}) - (-1)^{\ell+1}$$

$$0 = -(1-1)^\ell + \mu(p_1\ldots p_{\ell+1}) - (-1)^{\ell+1}$$

Hence $\mu(p_1\ldots p_{\ell+1}) = (-1)^{\ell+1}$.

c. Following the hint, we let m contain all the prime factors of n but repeated only once. Then we have that

$$0 = \sum_{k|n} \mu(k)$$

$$0 = \sum_{k|m} \mu(k) + \sum_{k|n:p_i^2|k} \mu(k)$$

where the second sum is over all k which divide n and have a square divisor. We know the first sum to be 0, leaving

$$0 = \sum_{k|n:p_i^2|k} \mu(k)$$

Since all k in this sum contain a repeated prime factor, and this holds regardless of which combination we choose, we must have that $\mu(k) = 0$ for all these k. Since n is in this sum, $\mu(n) = 0$. Another way to think of this is that letting every term be 0 works, and the uniqueness of μ means that this is the only possibility.

This simply gives

$$\mu(n) = \begin{cases} (-1)^\ell & n \text{ is the product of } \ell \text{ distinct primes} \\ 0 & n \text{ has a repeated prime factor} \end{cases}$$

d. We have that $\mathbf{1} * f = n^2$, so convolving by μ gives $f = \mu * n^2$. Thus,

$$f(2^4 3^4) = \sum_{k|2^4 3^4} k^2 \mu(2^4 3^4/k)$$

Since the only factors of $2^4 3^4$ with $\mu \neq 0$ are $1, 2, 3, 6$, this gives

$$f(2^4 3^4) = 2^8 3^8 - 2^6 3^8 - 2^8 3^6 + 2^6 3^6$$

$$f(2^4 3^4) = 2^9 3^7$$

4. a. Since the elements of U_s have $f(1) \neq 1$ except for $f = \epsilon$, it is disjoint from U_m and U_a excluding ϵ. If a function f from U_m is in U_a, then since $f(p_1^{k_1} p_2^{k_2} \dots p_l^{k_l}) = f(p_1^{k_1}) \dots f(p_l^{k_l})$, we have $f(n) = 0$ for all $n > 1$; thus, f is the identity. Therefore all three sets are disjoint excluding ϵ.

b. Define $g(p_1^{k_1} \dots p_l^{k_l}) = \prod_{i=1}^k f(p_i^k)$. Then clearly $g(mn) = g(m)g(n)$ for relatively prime m, n, and $g(1) = 1$ since an empty product is 1. Therefore $g \in U_m$. Now consider $h = g^{-1} * f$. We get $g^{-1}(1) = 1$, so $h(1) = 1$. Furthermore, for any k and prime p, $(g^{-1} * f)(p^k) = \sum_{i=0}^k g^{-1}(p^i) f(p^{k-i}) = \sum_{i=0}^k g^{-1}(p^i) g(p^{k-i})$ by the definition of g and g^{-1}. However, this is just $(g^{-1} * g)(p^k) = 0$ by the property of inverses. Therefore $f = g * h$ is the convolution of a multiplicative and an anti-multiplicative function.

c. Any function f in U is just a scalar times some function h in U that satisfies $h(1) = 1$, and a scalar times h, say rh, is just $h * r\epsilon$. If we let $r = f(1)$ and $g_s = r\epsilon$, we get that $g_s^{-1} * f = h$ is in U and $(g_s^{-1} * f)(1) = h(1) = 1$, so by the previous part, we can write $g_s^{-1} * f = g_m * g_a$. Then $f = g_s * g_m * g_a$.

d. As before, define $G(p_1^{k_1} \dots p_l^{k_\ell}) = \prod_{i=1}^k F(p_i^k)$. Clearly $F(2^k) = 2$ and $F(p^k) = 1$ for any other prime p. Then $G(n)$ is just 2 if n is divisible by 2, and 1 otherwise; in other words, $G(n) = \gcd(2, n)$. Now look at the third case for F. If n is not divisible by 2, this is just the number of pairs of prime factors of n, which suggests that H may be 1 for any number which is the product of two distinct prime factors. Trying this out reveals that it works; $F = \gcd(2, n) * H$ where

$$H(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n \text{ is the product of two distinct primes} \\ 0 & \text{otherwise} \end{cases}$$