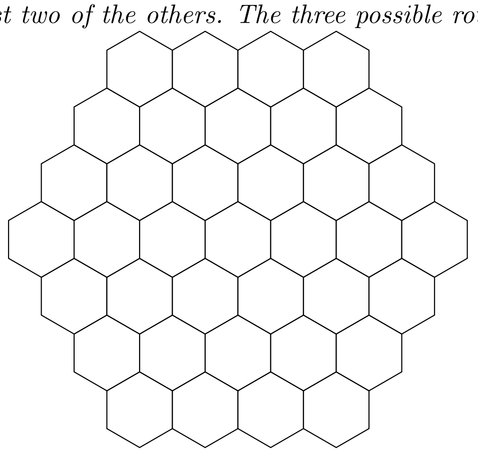
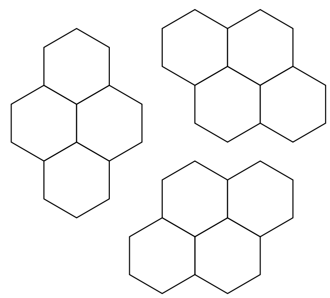
Team Round

2014 CHMMC

Problem 1. *Suppose we have a hexagonal grid in the shape of a hexagon of side length 4 as shown at left. Define a "chunk" to be four tiles, two of which are adjacent to the other three, and the other two of which are adjacent to just two of the others. The three possible rotations of these are shown at right.*

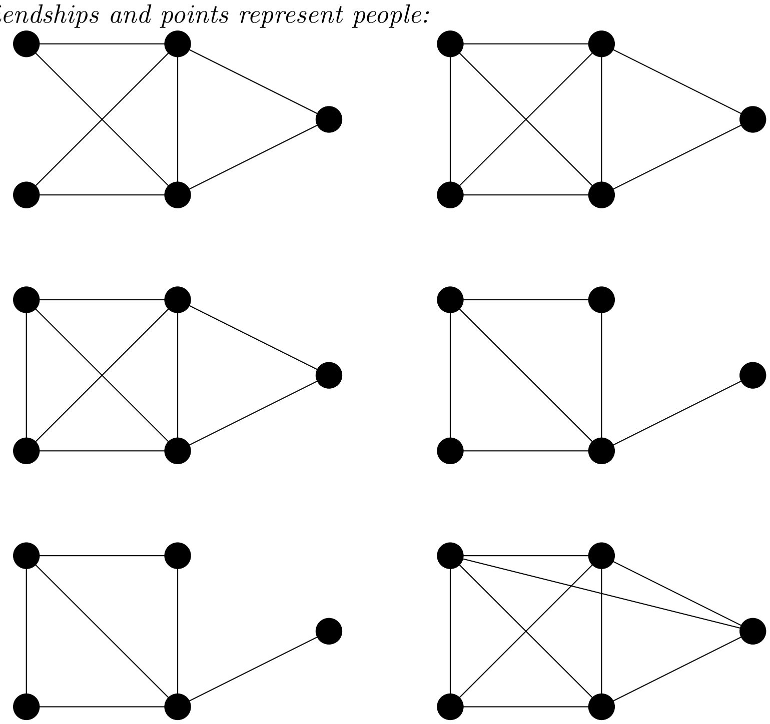




In how many ways can we choose a chunk from the grid?

Problem 2. *Consider two overlapping regular tetrahedrons of side length 2 in space. They are centered at the same point, and the second one is oriented so that the lines from its center to its vertices are perpendicular to the faces of the first tetrahedron. What is the volume encompassed by the combined solid?*

Problem 3. *Suppose that in a group of 6 people, if A is friends with B, then B is friends with A. If each of the 6 people draws a graph of the friendships between the other 5 people, we get these 6 graphs, where edges represent friendships and points represent people:*



If Sue drew the first graph, how many friends does she have?

Problem 4. *Let* $b_1 = 1$ *and* $b_{n+1} = 1 + \frac{1}{n(n+1)b_1 b_2 \ldots b_n}$ *for* $n \geq 1$*. Find* b_{12}*.*

Problem 5. *A teacher gives a multiple choice test to 15 students and that each student answered each question. Each question had 5 choices, but remarkably, no pair of students had more than 2 answers in common. What is the maximum number of questions that could have been on the quiz?*

Problem 6. *Suppose the transformation* T *acts on points in the plane like this:*

$$T(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

Determine the area enclosed by the set of points of the form $T(x, y)$*, where* (x, y) *is a point on the edge of a length-2 square centered at the origin with sides parallel to the axes.*

Problem 7. *Let*

$$P(x) = \prod_{k=1}^{n} (x^{3^k} + x^{-3^k} - 1), \quad Q(x) = \prod_{k=1}^{n} (x^{3^k} + x^{-3^k} + 1).$$

Given that

$$P(x)Q(x) = \sum_{k=-2 \cdot 3^n}^{2 \cdot 3^n} a_k x^k,$$

Compute $\displaystyle\sum_{k=0}^{3^n} a_k$ *in terms of* n*.*

Problem 8. *What's the greatest pyramid volume one can form using edges of length* $2, 3, 3, 4, 5, 5$*, respectively?*

Problem 9. *There is a long-standing conjecture that there is no number with* $2n + 1$ *instances in Pascal's triangle for* $n \geq 2$*. Assuming this is true, for how many* $n \leq 100,000$ *are there exactly 3 instances of* n *in Pascal's triangle?*

Problem 10. *Consider a grid of all lattice points* (m, n) *with* m, n *between 1 and 125. There exists a "path" between two lattice points* (m_1, n_1) *and* (m_2, n_2) *on the grid if* $m_1 n_1 = m_2 n_2$ *or if* $m_1/n_1 = m_2/n_2$*. For how many lattice points* (m, n) *on the grid is there a sequence of paths that goes from* $(1, 1)$ *to* (m, n)*?*