Tiebreaker Round

2014 CHMMC

Problem 1. *For* $a_1, \ldots a_5 \in \mathbb{R}$,

$$\frac{a_1}{k^2+1} + \ldots + \frac{a_5}{k^2+5} = \frac{1}{k^2}$$

for all $k \in \{2, 3, 4, 5, 6\}$. *Calculate*

$$\frac{a_1}{2} + \ldots + \frac{a_5}{6}$$

Problem 2. *A matrix* $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ *has square root* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *if*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Determine how many square roots the matrix $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ *has (complex coefficients are allowed).*

Problem 3. *Two players play a game on a pile of* n *beans. On each players´ turn, they may take exactly* 1, 4, *or* 7 *beans from the pile. One player goes first, and then the players alternate until somebody wins. A player wins when they take the last bean from the pile. For how many* n *between 2014 and 2050 (inclusive) does the second player win?*

Problem 4. *If* $f(i, j, k) = f(i-1, j+k, 2i-1)$ *and* $f(0, j, k) = j + k$, *evaluate* $f(n, 0, 0)$.

Problem 5. *Determine the value of*

$$\prod_{n=1}^{\infty} 3^{n/3^n} = \sqrt[3]{3}\ \sqrt[3^2]{3^2}\ \sqrt[3^3]{3^3} \ldots$$