Individual Round

CHMMC 2016

November 20, 2016

Problem 1. *We say that* $d_k d_{k-1} \cdots d_1 d_0$ represents the number n in base -2 *if each* d_i *is either 0 or 1, and* $n = d_k(-2)^k + d_{k-1}(-2)^{k-1} + \cdots + d_1(-2) + d_0$*. For example,* 110 *represents the number* 2 *in base* -2*. What string represents* 2016 *in base* -2*?*

Problem 2. *Alice and Bob find themselves on a coordinate plane at time* $t = 0$ *at points* $A(1, 0)$*, and* $B(-1, 0)$*. They have no sense of direction, but they want to find each other. They each pick a direction with uniform random probability. Both Alice and Bob travel at speed* $1 \frac{unit}{min}$ *in their chosen directions. They continue on their straight line paths forever, each hoping to catch sight of the other. They each have a* 1*-unit radius field of view: they can see something iff its distance to them is at most* 1*. What is the probability that they will ever see each other?*

Problem 3. *A gambler offers you a* \$2 *ticket to play the following game. First, you pick a real number* $0 \le p \le 1$*. Then, you are given a weighted coin with probability* p *of coming up heads and probability* $1 - p$ *of coming up tails, and flip this coin twice. The first time the coin comes up heads, you receive* \$1*, and the first time it comes up tails, you receive* \$2*. Given an optimal choice of* p*, what is your expected net winning?*

Problem 4. *Compute*

$$\sum_{n \ge 1} \frac{2^{n+1}}{8 \cdot 4^n - 6 \cdot 2^n + 1}.$$

Problem 5. *Suppose you have* 27 *identical unit cubes, where on each cube the three faces adjacent to one vertex are colored red, and the three faces adjacent to the opposite vertex are colored blue. The cubes are assembled randomly into a single* 3 *by* 3 *by* 3 *cube. (In particular, the orientation of each unit cube is distributed uniformly over the possible orientations.) The probability that the outside of this cube is entirely a single color is equal to* $\frac{1}{2^n}$*. Find* n*.*

Problem 6. *How many binary strings of length* 10 *are there that don't contain either of the substrings* 101 *or* 010*?*

Problem 7. *Let* $f(x) = \frac{1}{1 - \frac{3x}{16}}$*. Consider the sequence* $0, f(0), f(f(0)), f^3(0), \ldots f^n(0), \ldots$*. Find the smallest* L *such that* $f^n(0) \le L$ *for all* n*. If no such* L *exists, write "none".*

Problem 8. *Define* $n \% d$ *as the remainder when* n *is divided by* d*, i.e.* $n \% d$ *is the number* r *with* $n = qd + r$ *such that* $0 \le r < |d|$*. What is the smallest positive integer* n*, not divisible by* 5*,* 7*,* 11*, or* 13*, for which* $n^2 \,\%\, 5 < n^2 \,\%\, 7 < n^2 \,\%\, 11 < n^2 \,\%\, 13$*?*

Problem 9. *In quadrilateral* $ABCD$*,* $AB = DB$ *and* $AD = BC$*. If* $m\angle ABD = 36^\circ$ *and* $m\angle BCD = 54^\circ$*, find* $m\angle ADC$ *in degrees.*

Problem 10. *For a positive integer* n*, let* $p(n)$ *be the number of prime divisors of* n*, counted with multiplicity, so for example,* $p(3) = 1$*,* $p(4) = p(6) = 2$*. Now define the sequence* $a_0, a_1, a_2, \ldots$ *by* $a_0 = 2$*, and for* $n \ge 0$*,* $a_{n+1} = 8^{p(a_n)} + 2$*. Compute*

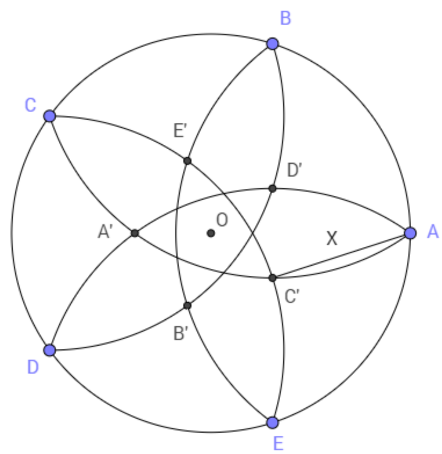
$$\sum_{n=0}^{\infty} \frac{a_n}{2^n}.$$

Problem 11. *Let $a, b \in [0, 1]$, $c \in [-1, 1]$ be chosen independently and uniformly at random. What is the probability that $p(x) = ax^2 + bx + c$ has a root in $[0, 1]$?*

Problem 12. *Let a be a positive real number, and let C be the cube with vertices $(\pm a, \pm a, \pm a)$ and T be the tetrahedron with vertices $(2a, 2a, 2a)$, $(2a, -2a, -2a)$, $(-2a, 2a, -2a)$, $(-2a, -2a, 2a)$. The intersection of C and T has volume ka^3 for some positive real k. What is k?*

Problem 13. *A sequence of numbers $a_1, a_2, \ldots, a_m$ is a geometric sequence modulo n of length m (for some positive integers n and m) if for each index i with $1 \leq i \leq m$ we have $a_i \in \{0, 1, 2, \ldots, n-1\}$ and there is some integer k such that n divides $(a_{j+1} - ka_j)$ for $j = 1, 2, \ldots, m-1$.*

How many geometric sequences modulo 14 of length 14 are there?



Problem 14. *Let circle O be a unit circle with five points, A, B, C, D, and E, spaced equidistantly along the circumference of the circle. For each of the points, there is an arc inside circle O with center at that point and beginning and ending at the two adjacent points (e.g., for point A, there is an arc of center A beginning at E and ending at B). The arcs intersect each other at points A', B', C', D', and E', as shown in the diagram. Find X, the length of $\overline{AC'}$. You may leave your answer in the form $f(x)$, where f is a trigonometric function and x is in simplest form.*

Problem 15. *How many pairs of nonintersecting closed rectangles are there in a 5 by 5 grid? (By "closed", we mean the rectangles include their boundaries, so for example, the pair on the right intersects, while the pair on the left does not).*