Power Round

CHMMC 2016

November 20, 2016

1 The Magic Square Game (24 pts)

Alice and Bob tell their friend Eve that they have a magic 3×3 square of numbers with the following properties:

- Every entry is either 1 or -1
- The product of each column is 1
- The product of each row is -1.

Problem 1.1. (5 pts) Exhibit such a square or prove that none exists.

Eve refuses to believe her friends, and they refuse to show their square to Eve. To resolve the dispute, they have the idea to play the following game.

Definition 1.1 (Magic Square Game)**.**

1. Eve randomly generates two numbers $x, y \in \{0, 1, 2\}$ independently and uniformly at random. She gives x to Alice and y to Bob.
2. Without communicating, Alice and Bob each produce a triple of ± 1 numbers (a_0, a_1, a_2) and (b_0, b_1, b_2).
3. Eve checks that $a_0 a_1 a_2 = 1$, $b_0 b_1 b_2 = -1$.
4. Eve checks that $a_y = b_x$.

Alice and Bob win if they pass both of Eve's checks. They lose if they fail either of them.

Problem 1.2. (6 pts) Alice, Bob, and Eve play three instances of the game. In each instance, decide whether Alice and Bob win or lose the game. Either show that they fail one of the checks or that they pass both.

1. Eve asks $x = 0$, $y = 1$. Alice answers $a = (1, 1, 1)$, while Bob answers $b = (-1, -1, -1)$.
2. Eve asks $x = 1$, $y = 0$. Alice answers $a = (1, -1, -1)$, while Bob answers $b = (-1, 1, -1)$.

3. Eve asks $x = 2$, $y = 2$. Alice answers $a = (1, -1, -1)$, while Bob answers $b = (1, 1, -1)$.

Problem 1.3. (8 pts) Give a deterministic strategy that wins with probability $\frac{8}{9}$.

Problem 1.4. (5 pts) Suppose for the sake of this problem that a square of numbers

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix}$$

satisfying the conditions of problem 1.1 does exist (regardless of your answer to problem 1.1), and that Alice and Bob have access to this square. Show how they can use this square to make a winning strategy in Eve's game.

Eve is happy because she thinks that if they play this game many times and Alice and Bob always win, then they must have the square. Alice and Bob are happy because they only have to reveal part of their square at a time. In order to avoid letting Eve learn the square, they can change which square they use each time they play the game. In section 3, you'll see that in a classical theory of physics, Eve's intuition is correct. In section 2, you'll see that with quantum mechanics, Alice and Bob can fool Eve.

2 A Quantum Strategy for the Magic Square Game (49 pts)

Now we'll show that if we allow Alice and Bob to take advantage of quantum mechanical phenomena, they have a winning strategy for the Magic Square game. First, we'll need some properties of the *Pauli group*.

2.1 The Pauli Group

Definition 2.1. The Pauli group on one qubit (which we'll now denote by $\mathcal{P}_1$) is a multiplicative structure consisting of the four Pauli operators I, X, Y, Z together with multiplicative constants $1, -1, i, -i$. Explicitly, the sixten elements of the Pauli group are

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\} \tag{1}$$

We say that elements A and B *commute* if $AB = BA$. We say they *anti-commute* if $AB = -BA$.

The Pauli group obeys the following relations:

- The Pauli group is *closed under multiplication*, i.e. every product of two Pauli operators is a Pauli operator times a constant.
- Multiplication is associative. That is, $(AB)C = A(BC)$ for any $A, B, C \in \mathcal{P}_1$. We'll write ABC without ambiguity.
- I is the identity element. In other words, if A is any element of $\mathcal{P}_1$, then $IA = AI = A$.
- $X^2 = Y^2 = Z^2 = I$.
- $XYZ = -iI$.
- The multiplicative constants $\pm 1, \pm i$ act like they do in the complex numbers, i.e. $i^2 = -1$ and $(-1)^2 = 1$.
- The multiplicative constants commute with everything. (More formally, we should say that $\pm I$ and $\pm iI$ commute with everything.)

Problem 2.1. (9 pts) Prove that X, Y, and Z are *pairwise anti-commuting*. That is, for every pair $A, B \in \{X, Y, Z\}$ with $A \neq B$, we have $AB = -BA$.

Remark 2.2. There are four 2×2 matrices over the complex numbers that obey the same relations as the I, X, Y, Z given here. You are *not* asked to find them.

Definition 2.3. The *Pauli group on two qubits*, denoted $\mathcal{P}_2$, consists of pairs of elements from the Pauli group on one element, together with multiplicative constants $\{1, i, -1, -i\}$. We think of the first part of the pair as being a Pauli operator acting on the first qubit and and the second as being a Pauli operator acting on the second qubit. We write the pair using the tensor product notation

$\otimes$. Explicitly, every element of the Pauli group can be written like $cA \otimes B$, where A and B are Pauli operators and $c \in \{\pm 1, \pm i\}$, because multiplication obeys the following relations:

$$(i^n A) \otimes (i^m B) = i^{n+m}(A \otimes B); \quad (i^n A \otimes B)(i^m C \otimes D) = i^{n+m}(AC \otimes BD). \quad (2)$$

Some typical elements of $\mathcal{P}_2$ are $X \otimes X$, $-iI \otimes Z$, and $Y \otimes Z$. (Note that the notation $-iI \otimes Z$ is well-defined by the first equation above.)

Problem 2.2. (22 pts)

a) Show that $X \otimes X$ commutes with $Z \otimes Z$.

b) Compute the product $(X \otimes X)(Y \otimes Z)(Z \otimes Y)$ as a tensor of two one-qubit Pauli operators, possibly with a multiplicative constant.

c) Of the 16 two-qubit Pauli operators of the form $U \otimes V$, where $U, V \in \{I, X, Y, Z\}$, how many commute with $X \otimes X$? How many anticommute? Give proof for your answer.

Problem 2.3. (13 pts) Find, with proof, a 3×3 square of two-qubit Pauli operators such that:

- In each row, the three operators pairwise commute and their product is $-I \otimes I$.
- In each column, the three operators pairwise commute and their product is $I \otimes I$.

(Hint: There is such a square with the property that every single-qubit operator (including I) appears at least once. You may need to include -1 coefficients.)

Problem 2.4. (5 pts) Why doesn't your proof from Problem 1.1 apply to the square you found in Problem 2.3?

Theorem 2.4. *If Alice and Bob have access to quantum-mechanical devices, they can use the magic square found in Problem 2.3 to win the Magic Square Game with certainty.*

(A treatment of this theorem can be found at `https://pdfs.semanticscholar.org/33bf/805817648a88f06707dde3e627bfdd74945a.pdf`)

3 A Bell Inequality (27 pts)

First, we need to formalize the notion of a game. Let $X = Y = \{0, 1, 2\}$ and let

$$A = B = \{(+1,+1,+1),(+1,+1,-1),(+1,-1,+1),\\ (+1,-1,-1),(-1,+1,+1),(-1,+1,-1),(-1,-1,+1),(-1,-1,-1)\}. \quad (3)$$

We'll think of X and Y as sets of questions by Eve and A and B as the set of valid answers by Alice and Bob.

Definition 3.1. A *strategy* for Alice and Bob is a function $p : A \times B \times X \times Y \to [0, 1]$ such that for each fixed x, y, $\sum_{ab} p(a, b\|x, y) = 1$.

If Eve asks question x to Alice and question y to Bob, then the probability that Alice produces answer a while Bob produces answer b is $p(a, b\|x, y)$. (The symbol $\|$ should be read as "given" or "conditioned on".) Let $V : A \times B \times X \times Y \to \{0, 1\}$ be the *valuation* for the game, i.e. the function that tells whether Alice and Bob win the game. That is,

$$V(a,b,x,y) = \begin{cases} 1, & \text{if } a_y = b_x,\ a_0a_1a_2 = 1, \text{ and } b_0b_1b_2 = -1 \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

Problem 3.1. (12 pts) Show that the probability that Alice and Bob win the game is given by

$$\omega(p) := \frac{1}{9} \sum_{\substack{a\in A, b\in B \\ x\in X, y\in Y}} p(a, b\|x, y)V(a, b, x, y). \quad (5)$$

(Remember that Eve picks x and y independently and uniformly at random.)

Definition 3.2. A strategy is *local* if it decomposes as a product of one strategy for Alice and one strategy for Bob. Explicitly, p is local if there exist $p_A : A \times X \to [0, 1]$ and $p_B : B \times Y \to [0, 1]$ such that

$$p(a, b\|x, y) = p_A(a, x) \cdot p_B(b, y). \quad (6)$$

A strategy is *classical* if it is a convex combination of local strategies. Explicitly, p is classical if there exist local strategies $p_1, \dots, p_n$ and nonnegative real numbers $c_1 \dots c_n$ such that $\sum_{i=1}^n c_i = 1$ and such that for all a, b, x, y,

$$p(a, b\|x, y) = \sum_{i=1}^{n} c_i p_i(a, b\|x, y). \quad (7)$$

Intuitively, we say that a strategy is classical if Alice and Bob can implement it by making use of classically correlated[1] random variables. For example, they might both look at the weather reports for Pasadena and choose to make their first bits equal to 0 if it's sunny, or make them equal to 1 if it's *extra* sunny.

[1] Here, *classically correlated* means something like "obeying a naïve interpretation of physics widely regarded as an accurate model of the real world before the discovery of quantum mechanics in the 1930s." For more precise definitions and theorems, see Bell's seminal paper.

Problem 3.2. (15 pts) Prove that no classical strategy achieves a win probability strictly greater than $\frac{8}{9}$. In other words, assuming that p is classical, prove the following *Bell inequality:* $\omega(p) \leq \frac{8}{9}$, where ω is as defined in equation (5).