Team Solutions

November 19, 2017

1. Let p, q, r, and s be four distinct primes such that $p+q+r+s$ is prime, and the numbers p^2+qr and p^2+qs are both perfect squares. What is the value of $p+q+r+s$?

   Proposed by Vinayak Kumar

   Answer
   $\boxed{23}$

   Solution
   Since the sum $p+q+r+s$ is a prime greater than 2, the sum is odd.

   It follows that exactly one of the four primes is 2.

   If we had $2 \in \{q,r,s\}$ then it would mean that p^2+2k would be a perfect square for some odd k. However, such an expression is $3 \pmod 4$, and thus can never be a perfect square.

   Hence we must have $p=2$.

   Now without loss of generality, suppose $r<s$ and let $4+qr=m^2$ and $4+qs=n^2$ for even integers $m<n$.

   Subtracting 4 from both sides of each of these equations yields

   $$qr=(m+2)(m-2)$$

   and

   $$qs=(n+2)(n-2).$$

   Since none of the factors on the RHS of the equations can be 1, it follows $q,r \in \{m-2,m+2\}$ and $q,s \in \{n-2,n+2\}$. Because $m-2<n-2<n+2$ and $m+2<n+2$ it follows $q=m+2=n-2$.

   Then $qr=q(q-4)$ and $qs=q(q+4)$, in which it follows $r=q-4$ and $s=q+4$. We wish $q-4,q,q+4$ to be prime numbers. Since these three numbers have different residues modulo 3, the smallest of them, $q-4$, must be equal to 3. Hence $(r,q,s)=(3,7,11)$.

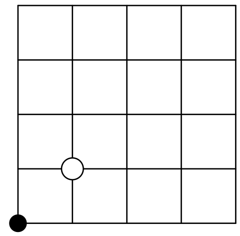
   Coupling this with the fact that $p=2$ we deduce that

   $$p+q+r+s=2+7+3+11=23.$$

2. Adam the spider (the black dot • in the figure) is sitting at the bottom left of a 4×4 coordinate grid, where adjacent parallel grid lines are each separated by one unit. He wants to crawl to the top right corner of the square, and starts off with 9 "crumb's" worth of energy.

   Adam only walks in one-unit segments along the grid lines, and cannot walk off of the grid. Walking one unit costs him one crumb's worth of energy, and Adam cannot move anymore once he runs out of energy. Also, Adam stops moving once he reaches the top right corner.

There is also a single crumb (the white dot $\circ$ in the figure) on the grid located one unit to the right and one unit up from Adam's starting position. If he goes to this point and eats the crumb, he will gain one crumb's worth of energy.



How many paths can Adam take to get to the upper right corner of the grid? Note that Adam does not care if he has extra energy left over once he arrives at his destination.

Proposed by Arushi Gupta

Answer

$\boxed{1046}$

Solution

Label Adam's starting point as $(0,0)$ and the top right corner as $(4,4)$ in a section of the coordinate plane.

There are two cases to consider: either Adam gets to $(4,4)$ in exactly 8 steps, or he gets there in exactly 10 steps. In the latter case, he must eat the crumb on his way up to $(4,4)$.

By standard block walking arguments, there are $\binom{8}{4} = 70$ for Adam to get to $(4,4)$ in exactly 8 steps.

For the second case, either Adam gets to $(1,1)$ in 4 steps and then gets to $(4,4)$ in 6 more steps, or he gets to $(1,1)$ in 2 steps and then gets to $(4,4)$ in 8 steps.

There are 8 ways to get to $(1,1)$ in 4 steps for the first time. There are then $\binom{6}{3}$ ways to get from $(1,1)$ to $(4,4)$. So overall there are $8\binom{6}{3} = 160$ ways in this case.

For the final case, there are 2 ways to get to $(1,1)$ in exactly two steps. To then get from $(1,1)$ to $(4,4)$ using 8 steps, we need to take either 3 steps up, 4 steps right, and 1 step left in some order, or 3 steps right, 4 steps up, and 1 step down in some order. By symmetry, we can just count the first scenario and then multiply by 2.

There are $\binom{8}{3}$ ways to determine which of Adam's steps are steps up. There are 4 choices of which step Adam should take to the left (since the step to the left must occur before the final step to the right). However, we have to subtract off the $\binom{6}{3}$ paths that end with Adam moving left and then right at the very end (since this corresponds to reaching $(4,4)$ for the first time in a total of 8 steps).

So overall we have

$$2 \cdot 2\left(4\binom{8}{3} - \binom{6}{3}\right) = 816$$

possible paths in this case.

Hence Adam has

$$70 + 160 + 816 = 1046$$

possible paths.

3. Two towns, A and B, are 100 miles apart. Every 20 minutes (starting at midnight) a bus traveling at 60 mph leaves town A for town B, and every 30 minutes (starting at midnight) a bus traveling at 20 mph leaves town B for town A. Dirk starts in Town A and gets on a bus leaving for town B at noon. However, Dirk is always afraid he has boarded a bus going in the wrong direction, so each time the bus he is in passes another bus, he gets out and transfers to that other bus. How many hours pass before Dirk finally reaches Town B?

Proposed by Gideon Leeper

Answer

$\boxed{14}$

Solution

We merely outline the general approach of the solution.

If we draw the lines representing the position of each bus as a function of time, and consider intersection points between these lines as spots where Dirk switches buses, we can show that for any positive integer n, Dirk will first reach the point $10n$ miles from Town B after $\frac{3n}{2} - 1$ hours.

It follows that 14 hours will pass before Dirk first reaches Town B.

4. Let $a = e^{4\pi i/5}$ be a nonreal fifth root of unity and $b = e^{2\pi i/17}$ be a nonreal seventeenth root of unity. Compute the value of the product

$$(a+b)\,(a+b^{16})(a^2+b^2)(a^2+b^{15})(a^3+b^8)(a^3+b^9)(a^4+b^4)(a^4+b^{13}).$$

Proposed by Haoyuan Sun

Answer

$\boxed{1}$

Solution

By looking at powers of 2 modulo 5 and modulo 17, we can rewrite the given product as

$$P = (a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})(a^{32}+b^{32})(a^{64}+b^{64})(a^{128}+b^{128})$$
$$= \prod_{j=0}^{7}\left(a^{2^j}+b^{2^j}\right).$$

Using difference of squares, we can write $a^{2^j}+b^{2^j} = \left(a^{2^{j+1}} - b^{2^{j+1}}\right) / \left(a^{2^j} - b^{2^j}\right)$ and telescope the above product down to

$$P = \prod_{j=0}^{7} \frac{a^{2^{j+1}} - b^{2^{j+1}}}{a^{2^j} - b^{2^j}} = \frac{a^{256} - b^{256}}{a-b}.$$

However, $a^{256} = e^{1024\pi i/5} = e^{4\pi i/5} = a$ since $256 \cdot 4 \equiv 4 \pmod 1)0$.

An analogous calculation shows that $b^{256} = b$.

Hence the given product P just equals 1.

5. Felix picks four points uniformly at random inside a unit circle $\mathcal{C}$. He then draws the four possible triangles which can be formed using these points as vertices. Finally, he randomly chooses one of the six possible pairs of the triangles he just drew.

   What is the probability that the center of the circle $\mathcal{C}$ is contained in the union of the interiors of the two triangles that Felix chose?

   Proposed by Felix Weilacher

   Answer

   $\boxed{5/12}$

   Solution

   We first calculate the probability that any of the four triangles contain the center of the circle. Sliding points along their radius does not alter containment of the center, so we can think of the points as lying on the circumference of the circle.

   The union of the four triangles Felix draws is the convex hull of the four chosen points points. This union fails to contain the origin if and only if all four points are contained in a single semicircular arc. This holds if and only if for one of the points, the other three are contained in the arc of this length obtained by starting at this point and moving clockwise, and this condition can hold for at most one of the four points, with measure 0 exceptions. Given a point, the probability the other three lie on this clockwise arc is just $(1/2)^3 = 1/8$, so since these events are disjoint, the probability that the center is not in our convex hull is $4 \cdot (1/8) = 1/2$.

   Now, any point in the plane not on the sides of any of these triangles is either contained in none of the triangles, or two of them. Therefore (ignoring cases that show up with probability zero) the center is contained in the union of the four triangles above if and only if it is contained in the intersection of some two of them, and these possibilities are all disjoint.

   Since there are 6 pairs, the probability of the point being contained in the intersection of some pair is $(1/2)/6 = 1/12$. The point is contained in our union of two triangles if and only if it is contained in the union of all four, but not contained in the intersection of two triangles which we did not select. Therefore our answer is $1/2 - 1/12 = 5/12$.

6. The country of Claredena has 5 cities, and is planning to build a road system so that each of its cities has exactly one outgoing (unidirectional) road to another city.

   Two road systems are considered equivalent if we can get from one road system the other by just changing the names of the cities. That is, two road systems are considered the same if given a relabeling of the cities, if in the first configuration a road went from city C to city D, then in the second configuration there is road that goes from the city now labeled C to the city now labeled D.

   How many distinct, nonequivalent possibilities are there for the road system Claredena builds?

   Proposed by Mayank Pandey

   Answer

   $\boxed{13}$

   Solution

   Consider the cities as vertices and the roads as edges of a directed graph G. Since each city has exactly one outgoing road, G is a graph where each connected component is a cycle of trees. We count the number of nonequivalent possibilities for G by casework on the number of connected components, and the lengths of the cycles that make up those components.

If G has only one connected component, the cycle that makes it up has between 2 and 5 vertices.

If there are 5 vertices in the cycle, there is only one possible graph.

If there are 4 vertices in the cycle, there is only one possible graph.

If there are 3 vertices in the cycle, then either the multiset of the number of vertices on the three trees corresponding to each of the nodes is $\{1,1,3\}$ or $\{1,2,2\}$. In the first case, there are 2 such graphs, since either there are two vertices that both point to the vertex in the cycle, or the tree is just a path. In the case of $\{1,2,2,\}$, there is only 1 graph. Therefore, when the cycle has 3 vertices, there are 3 possible graphs.

If there are 2 vertices in the cycle, then either the number of vertices on the trees are $\{1,4\}$ or $\{2,3\}$. The case $\{1,4\}$ has 4 graphs as there are 4 non-equivalent rooted trees with 4 vertices. The case $\{2,3\}$ has 2 trees since the tree with 2 vertices is unique, and there are two 3 vertex rooted trees.

Overall then, if G has one connected component there are

$$1+1+3+4+2=11$$

possible graphs.

If there are multiple connected components, there tree must have exactly two component, one of which has 2 vertices, and the other with 3 vertices. There are 2 possibilities for the 3 vertex component, and the 2 vertex component must be a 2-cycle.

Therefore, the answer should be $11+2=13$

7. Triangle ABC has side lengths $AB = 18, BC = 36$, and $CA = 24$. The circle Γ passes through point C and is tangent to segment AB at point A.

   Let X, distinct from C, be the second intersection point of Γ with segment BC. Moreover, let Y be the point on Γ such that segment AY is an angle bisector of $\angle XAC$.

   Suppose the length of segment AY can be written in the form

$$AY = \frac{p\sqrt{r}}{q}$$

   where p, q, and r are positive integers such that $\gcd(p,q) = 1$ and r is square free.

   Find the value of $p+q+r$.

   Proposed by Shyan Akmal

Answer
$\boxed{69}$

Solution
Since AB is tangent to Γ at point A, we get that $\angle BAX = \angle BCA$.

It follows by AA similarity that $ABC \sim XBA$. Since $BC = 2 \cdot AB$ it follows that the ratio of similitude from ABC to XBA is 2.

So we can compute $BX = AB/2 = 9$ and $AX = CA/2 = 18$.
Let D denote the intersection of AY with BC. Then using the angle bisector theorem we can compute $DX = 9$ and $DC = 18$.
Now set $d = AD$ and $y = AY$. By Power of a Point on D with respect to Γ, we have

$$d(y - d) = 9 \cdot 18.$$

Since AY is an angle bisector, we know that $\angle XAY = \angle DAC$. Because $AXYC$ is cyclic, we know that $\angle ACD = \angle AYX$. It follows by AA similarity that $AXY \sim ADC$. From this similarity we deduce that

$$\frac{12}{d} = \frac{y}{24} \implies dy = 12 \cdot 24.$$

If we subtract the earlier equation from the above equation we get that

$$d^2 = 12 \cdot 24 - 9 \cdot 18 = 18 \cdot (16 - 9) = 18 \cdot 7.$$

which implies that

$$d = 3\sqrt{14}.$$

If we substitute this into the previous equation, we get that

$$y = \frac{12 \cdot 24}{3\sqrt{14}} = \frac{48\sqrt{14}}{7}.$$

Thus $p + q + r = 48 + 7 + 14 = 69$.

8. Let $P(x)$ be the polynomial of degree at most 6 which satisfies $P(k) = k!$ for $k = 0, 1, 2, 3, \ldots, 6$.
Compute the value of $P(7)$.
Proposed by Shyan Akmal

Answer
$\boxed{3186}$

Solution
Take f_n to be the polynomial of degree at most n satisfying $f_n(k) = k!$ for $k = 0, 1, 2, \ldots, n$. Further define $T_n = f_n(n+1)$ for $n \geq 0$. To solve this problem, we need to compute T_6.
Now consider the polynomial

$$g(x) = f_n(x) - x f_{n-1}(x-1).$$

It follows that

$$g(x) = c \cdot (x-1)(x-2)\cdots(x-n)$$

for some constant c.
Since $g(0) = f_n(0) = 1$, if we evaluate both sides of the above equation at $x = 1$ we get that $c = (-1)^n/n!$ is forced. Thus

$$f_n(x) - x f_{n-1}(x-1) = \frac{(-1)^n}{n!} \cdot (x-1)(x-2)\cdots(x-n).$$

Evaluating both sides of the above equation at $x = n + 1$ yields the recurrence

$$T_n = (-1)^n + (n+1)T_{n-1}. \tag{$*$}$$

Since $f_1(x) = 1$, we know that $T_1 = 1$.
Now by repeatedly applying $(*)$, we can successively compute

$$\begin{aligned} T_2 &= 3T_1 + 1 = 4 \\ T_3 &= 4T_2 - 1 = 15 \\ T_4 &= 5T_3 + 1 = 76 \\ T_5 &= 6T_4 - 1 = 455 \\ T_6 &= 7T_5 + 1 = 3186 \end{aligned}$$

to get that $P(7) = 3186$.
Note: We can solve the given recurrence explicitly to find that

$$T_n = (n+1)! - D_{n+1}$$

where D_{n+1} is the number of derangements on $(n+1)$ elements.
This closed form can also be derived by applying Lagrange Interpolation.

9. Rachel the unicorn lives on the numberline at the number 0. One day, Rachel decides she'd like to travel the world and visit the numbers $1, 2, 3, \ldots, 31$.

   She starts off at the number 0, with a list of the numbers she wants to visit: $1, 2, 3, \ldots, 31$.

   Rachel then picks one of the numbers on her list uniformly at random, crosses it off the list, and travels to that number in a straight line path.

   She repeats this process until she has crossed off and visited all thirty-one of the numbers from her original list. At the end of her trip, she returns to her home at 0.

   What is the expected length of Rachel's round trip?

   Proposed by Shyan Akmal

Answer
$\boxed{352}$

Solution
More generally, suppose that Rachel wants to visit the numbers $1, 2, \ldots, n-1$ for some positive integer n. This problem is the special case where $n = 32$.
Let X be the random variable with value equal to the length of Rachel's round trip.
For each ordered pair of nonnegative integers (i, j) with $0 \le i, j \le n-1$ consider the random variable $X_{i,j}$ defined as

$$X_{i,j} = \begin{cases} |j - i| & \text{if Rachel travels from } i \text{ directly to } j \\ 0 & \text{otherwise.} \end{cases}$$

This variable acts as a scaled indicator function that just represents the distance traveled if Rachel moves directly from i to j.

By definition

$$X = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} X_{i,j}.$$

It follows by linearity of expectation

$$E[X] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} E\left[X_{i,j}\right].$$

Now, the probability that Rachel moves directly from i to j for any $i \neq j$ is just $\frac{1}{n-1}$. This is because given that Rachel is currently at i, by symmetry she is equally likely to visit any of the $n-1$ numbers $j \neq i$, for $0 \leq j \leq n-1$.
It follows that

$$E\left[X_{i,j}\right] = \frac{|j-i|}{n-1}.$$

Substituting this into the earlier equation we have

$$E[X] = \frac{1}{n-1} \cdot \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |j-i|.$$

Now, using the fact that $\ell = 0, 1, \ldots, n-1$ there are exactly $2(n-\ell)$ pairs (i, j) satisfying $|j-i| = \ell$, we can rewrite the sum on the right hand side of the above equation as

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |j-i| = 2 \cdot \sum_{\ell=0}^{n-1} \ell(n-\ell).$$

There are multiple ways of evaluating the sum on the right hand side. Below we present one approach, that involves interpreting the sum combinatorially.
For each nonnegative integer $\ell \leq n-1$, we can observe that ℓ is the number of nonnegative integer solutions (a, b) to

$$a + b = \ell - 1$$

and $(n-\ell)$ is the number of nonnegative integer solutions (c, d) to

$$c + d = n - \ell - 1.$$

It follows that $\sum_{\ell=0}^{n-1} \ell(n-\ell)$ is the number of nonnegative integer solutions (a, b, c, d) to

$$a + b + c + d = n - 2.$$

However, by a standard stars and bars argument, this quantity is equal to

$$\binom{n+1}{3}.$$

It follows that

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}|j-i| = 2\binom{n+1}{3}$$

which means that

$$E[X] = \frac{2}{n-1}\cdot\binom{n+1}{3} = \frac{n(n+1)}{3}.$$

For this particular problem we have $n = 32$, and thus the expected length of Rachel's round trip is

$$\frac{32\cdot 33}{3} = 33\cdot 11 = 352.$$

10. Let α be the unique real root of the polynomial $x^3 - 2x^2 + x - 1$. It is known that $1 < \alpha < 2$.

    We define the sequence of polynomials $\{p_n(x)\}_{n\geq 0}$ by taking $p_0(x) = x$ and setting

$$p_{n+1}(x) = (p_n(x))^2 - \alpha$$

    for each integer $n \geq 0$.

    How many distinct real roots does $p_{10}(x)$ have?

    Proposed by Gideon Leeper

Answer

232

Solution

First observe that $p_1(x) = x^2 - \alpha$ has exactly two real roots $r < s$. If we consider the intervals $I_1 = (-\infty, r], I_2 = [r,s], I_3 = [s,\infty)$ determined by these roots, we can observe that

$$p_1(I_1) = p_1(I_3) = [0,\infty)$$

and

$$p_1(I_2) = [-\alpha, 0].$$

Similarly, if we look at the $p_2(x)$, we can check that it has exactly four real roots $a < b < c < d$. If we consider the intervals determined by these roots, we can check that

$$p_2((-\infty,a]) = p_2([d,\infty))[0,\infty)$$

$$p_2([a,b]) = p_2([c,d]) = [-\alpha,0]$$

$$p_2([b,c]) = [0,\alpha^2-\alpha].$$

With those examples out of the way, we will move onto working on the general case. The main idea of our solution will be to split $\mathbb{R}$ into these intervals determined by the roots of p_n. Actually, we will look at intervals where p_n is monotonic, since this ends up making calculations easier.

By looking at the graphs of p_1, p_2, p_3, we observe that it suffices to consider intervals I for which the image of I under p_n is either $[0,\infty)$, $[-\alpha,0]$, or $[0,\alpha^2-\alpha]$. The reason this ends up happening is because we are told in the problem statement that

$$\left(\alpha^2-\alpha\right)^2 = \alpha^2,$$

and this conditions means that p_3 ends up having a double root at $x = 0$.

Note that for an interval I on which p_n is monotonic, if $p_n(I) = [0, \infty)$, then for p_{n+1} we can divide I into two monotonic intervals, one with image $[0, \infty)$, and one with image $[-\alpha, 0]$. If $p_n(I) = [-\alpha, 0]$, we can again divide I into two monotonic intevals, one with image $[-\alpha, 0]$ and one with image $[0, \alpha^2 - \alpha]$. Finally, if $p_n(I) = [0, \alpha^2 - \alpha]$, then its image under p_{n+1} is $[-\alpha, 0]$. We can write this as

$$[0, \infty) \mapsto [0, \infty) + [-\alpha, 0]$$

$$[-\alpha, 0] \mapsto [-\alpha, 0] + [0, \alpha^2 - \alpha]$$
$$[0, \alpha^2 - \alpha] \mapsto [-\alpha, 0]$$

Now let x_n be the number of intervals I in our partition with $p_n(I) = [0, \infty)$, let y_n be the number with $p_n(I) = [-\alpha, 0]$, and let z_n be the number with $p_n(I) = [0, \alpha^2 - \alpha]$.

The above gives $x_{n+1} = x_n$, $y_{n+1} = x_n + y_n + z_n$, and $z_{n+1} = y_n$. Since $x_1 = 2$, $y_1 = 2$, $z_1 = 0$, we have $x_n = 2$ for all $n \geq 1$, and $y_n = y_{n-1} + y_{n-2} + 2$ for all $n \geq 3$. This is almost the Fibonacci recurrence, and since $y_1 = 2$, $y_2 = 4$, we can solve to give $y_n = 2F_{n+2} - 2$ for all $n \geq 1$.

Note that each interval in our partition contains exactly one root, and each root is contained in exactly two intervals in the partition, so the number of roots of p_n is

$$\frac{x_n + y_n + z_n}{2} = \frac{2 + (2F_{n+2} - 2) + (2F_{n+1} - 2)}{2} = F_{n+3} - 1$$

for all $n \geq 2$. Thus $F_{13} - 1 = 232$ is the number of real roots of $p_{10}(x)$.