Tiebreaker Solutions

November 19, 2017

1. Let a, b be the roots of the quadratic polynomial $Q(x) = x^2 + x + 1$, and let u, v be the roots of the quadratic polynomial $R(x) = 2x^2 + 7x + 1$.

   Suppose P is a cubic polynomial which satisfies the equations

$$\begin{cases} P(au) & = Q(u)R(a) \\ P(bu) & = Q(u)R(b) \\ P(av) & = Q(v)R(a) \\ P(bv) & = Q(v)R(b). \end{cases}$$

   If M and N are the coefficients of x^2 and x respectively in $P(x)$, what is the value of $M + N$?

   Proposed by Shyan Akmal

   Answer
   $\boxed{-25}$

   Solution
   Let $\omega = e^{2\pi i/3}$ be a nonreal third root of unity. Since ω and ω^2 are roots of Q, we can, without loss of generality, write $a = \omega$ and $b = \omega^2$.
   Now, consider the polynomial

$$f(x) = P(\omega x) - Q(x)R(\omega).$$

   It is a cubic whose leading coefficient is equal to the leading coefficient of P (since $\omega^3 = 1$). The given equations also imply that f has roots at u and v.
   Similar reasoning shows that the polynomial

$$g(x) = P(\omega^2 x) - Q(x)R(\omega^2)$$

   is a cubic with roots at u and v, whose leading coefficient equals the leading coefficient of P.
   Then if we subtract these two polynomials, we get a polynomial

$$h(x) = f(x) - g(x) = P(\omega x) - P(\omega^2 x) - \left(R(\omega) - R(\omega^2)\right) Q(x)$$

   which has degree at most 2, and has roots at u and v.
   Thus, $h(x) = kR(x)$ for some complex number k.
   Now, if we evaluate both sides of this equation at $x = 0$, we get that

$$R(\omega^2) - R(\omega) = h(0) = kR(0) = k.$$

It follows that

$$P(\omega x) - P(\omega^2 x) - \left(R(\omega) - R(\omega^2)\right) Q(x) = -\left(R(\omega) - R(\omega^2)\right) R(x).$$

We can rearrange this as

$$P(\omega x) - P(\omega^2 x) = \left(R(\omega) - R(\omega^2)\right) \left(Q(x) - R(x)\right).$$

We can compute $R(\omega) - R(\omega^2) = 5(\omega - \omega^2)$.
If we equate coefficients of x^2 in the above equation we get

$$M(\omega^2 - \omega) = 5(\omega^2 - \omega) \cdot (-6)$$
$$\implies M = -30.$$

Similarly, if we equate coefficients of x we get

$$N(\omega - \omega^2) = 5(\omega^2 - \omega) \cdot (-1)$$
$$\implies N = 5.$$

Thus $M + N = -25$.

2. Let N be the number of sequences $a_1, a_2, \ldots, a_{10}$ of ten positive integers such that

   (i) the value of each term of the sequence at most 30,

   (ii) the arithmetic mean of any three consecutive terms of the sequence is an integer, and

   (iii) the arithmetic mean of any five consecutive terms of the sequence is an integer.

   Compute $\sqrt{N}$.

   Proposed by Shyan Akmal

Answer
$\boxed{2400}$

Solution
We start by picking arbitrary values for a_1 and a_2. This can be done in 30^2 ways.
Once we pick the value of a_1 and a_2, the value of a_3 is uniquely determined modulo 3. So we have 10 possible choices for the value of a_3.
Similarly, once we have picked values for a_2 and a_3, the value of a_4 is uniquely determined modulo 3, so we have 10 choices for this value as well.
Then, for $n \geq 5$, once we have fixed values for $a_{n-1}, a_{n-2}, a_{n-3}$, and a_{n-4}, the value of a_n is uniquely determined modulo 15 (by Chinese Remainder Theorem). Thus there are 2 possible values for each of $a_5, a_6, \ldots, a_{10}$ if we set those values in the listed order.
Altogether then we get that there are

$$N = 30^2 \cdot 10^2 \cdot 2^6$$

sequences satisfying the given properties.
Hence $\sqrt{N} = 30 \cdot 10 \cdot 8 = 2400$.

3. You are playing a game called "Hovse."

   Initially you have the number 0 on a blackboard.

   If at any moment the number x is written on the board, you can either:

   - replace x with $3x+1$
   - replace x with $9x+1$
   - replace x with $27x+3$
   - or replace x with $\left\lfloor \frac{x}{3} \right\rfloor$.

   However, you are not allowed to write a number greater than 2017 on the board. How many positive numbers can you make with the game of "Hovse?"

   Proposed by Haoyuan Sun

   Answer
   $\boxed{71}$

   Solution
   Omitted

4. Jordan has an infinite geometric series of positive reals whose sum is equal to $2\sqrt{2}+2$.

   It turns out that if Jordan squares each term of his geometric series and adds up the resulting numbers, he get a sum equal to 4.

   If Jordan decides to take the fourth power of each term of his original geometric series and add up the resulting numbers, what sum will he get?

   Proposed by Shyan Akmal

   Answer
   $\boxed{16/3}$

   Solution
   Let the first term of series be a, and the common ratio of that series be r. We are given that

$$\frac{a}{1-r} = 2\sqrt{2}+2 \tag{1}$$

and

$$\frac{a^2}{1-r^2} = 4. \tag{2}$$

If we divide equation (2) by equation (1), we find that

$$\frac{a}{1+r} = \frac{4}{2(\sqrt{2}+1)} = 2(\sqrt{2}-1).$$

If we add the above equation to equation (1) we find that

$$\frac{2a}{1-r^2} = 4\sqrt{2}.$$

If we divide equation (2) by the above equation we get that

$$\frac{a}{2} = \frac{1}{\sqrt{2}}$$

so $a = \sqrt{2}$.

If we substitute this back into equation (1) we can solve for r and find that $r = 1/\sqrt{2}$.

Thus the sum of the fourth powers of the terms of the geometric series is

$$\frac{a^4}{1 - r^4} = \frac{4}{1 - \frac{1}{4}} = \frac{16}{3}.$$

5. Find the number of primes p such that $p! + 25p$ is a perfect square.

   Proposed by Zachary Chase

   Answer

   $\boxed{1}$

   Solution

   The given expression can be written in the form

$$p \cdot ((p-1)! + 25).$$

In order for this to be a perfect square, $(p-1)! + 25$ must be divisible by p. By Wilson's theorem, this happens if and only if $p \mid 24$.

Hence $p = 2$ or $p = 3$.

If $p = 2$ we have $p! + 25p = 52$ is not a perfect square.

If $p = 3$ then $p! + 25p = 81$ is a perfect square.

So there is only one prime that works.