Team Round

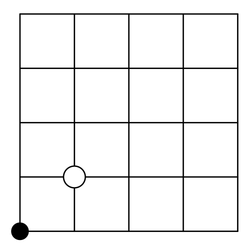
November 19, 2017

1. Let p, q, r, and s be four distinct primes such that $p+q+r+s$ is prime, and the numbers p^2+qr and p^2+qs are both perfect squares. What is the value of $p+q+r+s$?

2. Adam the spider (the black dot $\bullet$ in the figure) is sitting at the bottom left of a 4×4 coordinate grid, where adjacent parallel grid lines are each separated by one unit. He wants to crawl to the top right corner of the square, and starts off with 9 "crumb's" worth of energy.

   Adam only walks in one-unit segments along the grid lines, and cannot walk off of the grid. Walking one unit costs him one crumb's worth of energy, and Adam cannot move anymore once he runs out of energy. Also, Adam stops moving once he reaches the top right corner.

   There is also a single crumb (the white dot $\circ$ in the figure) on the grid located one unit to the right and one unit up from Adam's starting position. If he goes to this point and eats the crumb, he will gain one crumb's worth of energy.

   How many paths can Adam take to get to the upper right corner of the grid? Note that Adam does not care if he has extra energy left over once he arrives at his destination.

3. Two towns, A and B, are 100 miles apart. Every 20 minutes (starting at midnight) a bus traveling at 60 mph leaves town A for town B, and every 30 minutes (starting at midnight) a bus traveling at 20 mph leaves town B for town A. Dirk starts in Town A and gets on a bus leaving for town B at noon. However, Dirk is always afraid he has boarded a bus going in the wrong direction, so each time the bus he is in passes another bus, he gets out and transfers to that other bus. How many hours pass before Dirk finally reaches Town B?

4. Let $a = e^{4\pi i/5}$ be a nonreal fifth root of unity and $b = e^{2\pi i/17}$ be a nonreal seventeenth root of unity. Compute the value of the product
$$(a+b)\,(a+b^{16})(a^2+b^2)(a^2+b^{15})(a^3+b^8)(a^3+b^9)(a^4+b^4)(a^4+b^{13}).$$

5. Felix picks four points uniformly at random inside a unit circle $\mathcal{C}$. He then draws the four possible triangles which can be formed using these points as vertices. Finally, he randomly chooses of the six possible pairs of the triangles he just drew.

   What is the probability that the center of the circle $\mathcal{C}$ is contained in the union of the interiors of the two triangles that Felix chose?

6. The country of Claredena has 5 cities, and is planning to build a road system so that each of its cities has exactly one outgoing (unidirectional) road to another city.

   Two road systems are considered equivalent if we can get from one road system the other by just changing the names of the cities. That is, two road systems are considered the same if given a relabeling of the cities, if in the first configuration a road went from city C to city D, then in the second configuration there is road that goes from the city now labeled C to the city now labeled D.

   How many distinct, nonequivalent possibilities are there for the road system Claredena builds?

7. Triangle ABC has side lengths $AB = 18, BC = 36$, and $CA = 24$. The circle Γ passes through point C and is tangent to segment AB at point A.

   Let X, distinct from C, be the second intersection point of Γ with segment BC. Moreover, let Y be the point on Γ such that segment AY is an angle bisector of $\angle XAC$.

   Suppose the length of segment AY can be written in the form

$$AY = \frac{p\sqrt{r}}{q}$$

   where p, q, and r are positive integers such that $\gcd(p, q) = 1$ and r is square free.

   Find the value of $p + q + r$.

8. Let $P(x)$ be the polynomial of degree at most 6 which satisfies $P(k) = k!$ for $k = 0, 1, 2, 3, \ldots, 6$.

   Compute the value of $P(7)$.

9. Rachel the unicorn lives on the numberline at the number 0. One day, Rachel decides she'd like to travel the world and visit the numbers $1, 2, 3, \ldots, 31$.

   She starts off at the number 0, with a list of the numbers she wants to visit: $1, 2, 3, \ldots, 31$.

   Rachel then picks one of the numbers on her list uniformly at random, crosses it off the list, and travels to that number in a straight line path.

   She repeats this process until she has crossed off and visited all thirty-one of the numbers from her original list. At the end of her trip, she returns to her home at 0.

   What is the expected length of Rachel's round trip?

10. Let α be the unique real root of the polynomial $x^3 - 2x^2 + x - 1$. It is known that $1 < \alpha < 2$.

    We define the sequence of polynomials $\{p_n(x)\}_{n \geq 0}$ by taking $p_0(x) = x$ and setting

$$p_{n+1}(x) = (p_n(x))^2 - \alpha$$

    for each integer $n \geq 0$.

    How many distinct real roots does $p_{10}(x)$ have?