Tiebreaker Round

November 19, 2017

1. Let a, b be the roots of the quadratic polynomial $Q(x) = x^2 + x + 1$, and let u, v be the roots of the quadratic polynomial $R(x) = 2x^2 + 7x + 1$.

   Suppose P is a cubic polynomial which satisfies the equations

$$\begin{cases} P(au) &= Q(u)R(a) \\ P(bu) &= Q(u)R(b) \\ P(av) &= Q(v)R(a) \\ P(bv) &= Q(v)R(b). \end{cases}$$

   If M and N are the coefficients of x^2 and x respectively in $P(x)$, what is the value of $M + N$?

2. Let N be the number of sequences $a_1, a_2, \ldots, a_{10}$ of ten positive integers such that

   (i) the value of each term of the sequence at most 30,

   (ii) the arithmetic mean of any three consecutive terms of the sequence is an integer, and

   (iii) the arithmetic mean of any five consecutive terms of the sequence is an integer.

   Compute $\sqrt{N}$.

3. You are playing a game called "Hovse."

   Initially you have the number 0 on a blackboard.

   If at any moment the number x is written on the board, you can either:

   - replace x with $3x + 1$
   - replace x with $9x + 1$
   - replace x with $27x + 3$
   - or replace x with $\left\lfloor \frac{x}{3} \right\rfloor$.

   However, you are not allowed to write a number greater than 2017 on the board. How many positive numbers can you make with the game of "Hovse?"

4. Jordan has an infinite geometric series of positive reals whose sum is equal to $2\sqrt{2} + 2$.

   It turns out that if Jordan squares each term of his geometric series and adds up the resulting numbers, he get a sum equal to 4.

   If Jordan decides to take the fourth power of each term of his original geometric series and add up the resulting numbers, what sum will he get?

5. Find the number of primes p such that $p! + 25p$ is a perfect square.