Individual Round Solutions

2018

1. Two robots race on the plane from $(0,0)$ to (a,b), where a and b are positive real numbers with $a<b$. The robots move at the same constant speed. However, the first robot can only travel in directions parallel to the lines $x=0$ or $y=0$, while the second robot can only travel in directions parallel to the lines $y=x$ or $y=-x$. Both robots take the shortest possible path to (a,b) and arrive at the same time. Find the ratio $\frac{a}{b}$.

   Solution: Robot 1 will travel a distance $a+b$, while Robot 2 will travel a vertical distance b (since $b>a$) and thus total distance $b\sqrt{2}$. It follows that $a+b=b\sqrt{2}$, so $\frac{a}{b}=\boxed{\sqrt{2}-1}$.

2. Suppose $x+\frac{1}{x}+y+\frac{1}{y}=12$ and $x^2+\frac{1}{x^2}+y^2+\frac{1}{y^2}=70$. Compute $x^3+\frac{1}{x^3}+y^3+\frac{1}{y^3}$.

   Solution: Let $a=x+\frac{1}{x}$ and $b=y+\frac{1}{y}$. Then, $a^2-2=x^2+\frac{1}{x^2}$ and $b^2-2=y^2+\frac{1}{y^2}$. The equations give $a+b=12$ and $a^2+b^2-4=70$, which can be solved to give $a=5$ and $b=7$. Since $(x+\frac{1}{x})^3=x^3+3x+\frac{3}{x}+\frac{1}{x^3}$, we have $x^3+\frac{1}{x^3}=a^3-3a=110$. Similarly, $y^3+\frac{1}{y^3}=b^3-3b=322$, so the desired answer is $\boxed{432}$.

3. Find the largest non-negative integer a such that 2^a divides

$$3^{2^{2018}}+3.$$

   Solution: By Fermat's Little Theorem and the fact that $\phi(2^{2019})=2^{2018}$, we have

$$3^{2^{2018}}\equiv 1 \pmod{2^{2019}}.$$

   Hence the given number is congruent to $4 \pmod{2^{2019}}$, so $4=2^2$ is the largest power of 2 dividing it, and the largest value of a is $\boxed{2}$.

4. Suppose z and w are complex numbers, and $|z|=|w|=z\bar{w}+\bar{z}w=1$. Find the largest possible value of $\mathrm{Re}(z+w)$, the real part of $z+w$.

   Solution: Calculating $|z+w|^2$, we get $|z|^2+|w|^2+z\bar{w}+w\bar{z}=3$. Since $|z+w|=\boxed{\sqrt{3}}$, that is the maximum possible value of $\mathrm{Re}(z+w)$. (Alternatively, we can consider this problem geometrically and note that $z\bar{w}+\bar{z}w=2\cos\theta$, where θ is the angle between z and w in the complex plane.)

5. Two people, A and B, are playing a game with three piles of matches. In this game, a *move* consists of a player taking a positive number of matches from one of the three piles such that the number remaining in the pile is equal to the nonnegative difference of the numbers of matches in the other two piles. A and B each take turns making moves, with A making the first move. The last player able to make a move wins. Suppose that the three piles have 10, x, and 30 matches. Find the largest value of x for which A does not have a winning strategy.

Solution: Note that for three piles of a, b, and c matches, with $a \le b \le c$ and $a+b \le c$, the first move is predetermined; you must replace the pile of c matches with $b-a$ matches. In fact, since the new piles $b-a, a, b$ also satisfy this condition (perhaps in a different order), the next move is determined as well, and a brief induction shows that all following moves are predetermined if this initial condition is satisfied.

Note also that this situation can be effectively modeled by the Euclidean algorithm: if one always ignores the largest pile at any given time and focuses only on the smaller piles (which start with a and b matches), the number of matches left in the smaller two piles resembles another step in the Euclidean algorithm (there will be a and $b-a$ matches remaining, then a and $b-2a$, and so on until $b-ka < a$). A player wins when the algorithm finishes, i.e., when he or she removes all of the matches from one of the piles. This simplifies the computations, since you only need to keep track of two piles instead of three.

Thus for $x \le 20$, the game is completely predetermined, and it is easy to check that A loses when $x = 20$. For $x \ge 21$, A has a winning strategy: they can simply replace the pile of x matches with 20 matches. Thus the largest value of x for which A does not have a winning strategy is $\boxed{20}$.

6. Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon with side length 1. For $n = 1, \ldots, 6$, let B_n be a point on the segment A_nA_{n+1} chosen at random (where indices are taken mod 6, so $A_7 = A_1$). Find the expected area of the hexagon $B_1B_2B_3B_4B_5B_6$.

   Solution: For each n, the area of the triangle $B_{n-1}A_nB_n$ is $\frac{\sqrt{3}}{4}(B_{n-1}A_n)(A_nB_n)$, so since these two lengths are independent and each has expected value 1/2, the expected area of $B_{n-1}A_nB_n$ is $\frac{\sqrt{3}}{16}$. Then since $B_1B_2B_3B_4B_5B_6$ is $A_1A_2A_3A_4A_5A_6$ with these six triangles removed, it follows that the expected area of $B_1B_2B_3B_4B_5B_6$ is $\frac{3\sqrt{3}}{2} - 6(\frac{\sqrt{3}}{16}) = \boxed{9\sqrt{3}/8}$.

7. A termite sits at the point $(0,0,0)$, at the center of the octahedron $|x|+|y|+|z| \le 5$. The termite can only move a unit distance in either direction parallel to one of the x, y, or z axes: each step it takes moves it to an adjacent lattice point. How many distinct paths, consisting of 5 steps, can the termite use to reach the surface of the octahedron?

   Solution: The octahedron has 8 faces which the termite can reach. For each face, every path to that face consists of 5 independent choices from a set of three possible moves: $\{(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)\}$. So there are 3^5 ways to reach each face. However, we've double-counted every path which ends at an edge of the octahedron: there are 12 edges and 2^5 ways to reach each one. But in removing these, we've discounted every path which ends at a vertex of the octahedron, of which there are 6. Thus, our final result is

$$8 \cdot 3^5 - 12 \cdot 2^5 + 6 \cdot 1^5 = \boxed{1566}$$

   by Inclusion-Exclusion.

8. Let

$$P(x) = x^{4037} - 3 - 8 \cdot \sum_{n=1}^{2018} 3^{n-1}x^n.$$

   Find the number of roots z of $P(x)$ with $|z| > 1$, counting multiplicity.

Solution: Notice that $P(x) = (x-3)Q(x)$, where

$$Q(x) = x^{4036} + 3x^{4035} + \cdots + 3^{2018}x^{2018} + \cdots + 3x + 1.$$

Since Q is palindromic, $Q(x) = 0$ if and only if $Q(1/x) = 0$, and Q has no roots with absolute value 1, because for $|x| = 1$ the triangle inequality gives

$$|Q(x) - 3^{2018}x^{2018}| \leq 2\sum_{k=0}^{2017} 3^k = 3^{2018} - 1 < 3^{2018} = |3^{2018}x^{2018}|$$

meaning $Q(x) \neq 0$. It follows that P has exactly $1 + 4036/2 = \boxed{2019}$ roots with absolute value greater than 1.

9. How many times does 01101 appear as a not necessarily contiguous substring of 0101010101010101? (Stated another way, how many ways can we choose digits from the second string, such that when read in order, these digits read 01101?)

   Solution: Equivalently, the problem is to count the number of 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ with $1 \leq a_1 < a_2 < a_3 < a_4 < a_5 \leq 16$ such that a_1, a_4 are odd and a_2, a_3, a_5 are even. Now, look instead at the consecutive differences. For such a 5-tuple, $a_2 - a_1, a_4 - a_3, a_5 - a_4$ are odd, while $a_3 - a_2$ is even, so we can write $a_1 = 2b_1 + 1$, $a_2 = a_1 + 2b_2 + 1$, $a_3 = a_2 + 2b_3 + 2$, $a_4 = a_3 + 2b_4 + 1$, $a_5 = a_4 + 2b_5 + 1$, and $16 = a_5 + 2b_6$, for some integers $b_1, b_2, b_3, b_4, b_5, b_6 \geq 0$, which then must satisfy $16 = 2(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) + 6$, hence $b_1 + \cdots + b_6 = 5$. Conversely, for any $b_1, b_2, b_3, b_4, b_5, b_6 \geq 0$ with $b_1 + \cdots + b_6 = 5$, we can construct a corresponding 5-tuple $(a_1, a_2, a_3, a_4, a_5)$ satisfying the required properties, so our problem is exactly to count the number of such $(b_1, b_2, b_3, b_4, b_5, b_6)$. Stars and bars gives the answer $\binom{6+5-1}{5} = \boxed{252}$.

10. A *perfect number* is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive divisors excluding the number itself. For example, 28 is a perfect number because $1 + 2 + 4 + 7 + 14 = 28$. Let n_i denote the i^{th} smallest perfect number. Define

    $$f(x) = \sum_{i|n_x} \sum_{j|n_i} \frac{1}{j}$$

    (where $\sum_{i|n_x}$ means we sum over all positive integers i that are divisors of n_x). Compute $f(2)$, given there are at least 50 perfect numbers.

    Solution: For each n_i, we claim that $\sum_{j|n_i} 1/j = 2$. From this, it will follow that since $n_2 = 28$ and there are 6 divisors of 28, the sum is $2 \cdot 6 = \boxed{12}$. We can see that $\sum_{j|n_i} 1/j = 2$ by taking pairs of divisors whose product is n_i. Since $\frac{1}{j_1} + \frac{1}{j_2} = \frac{j_1+j_2}{j_1 j_2} = \frac{j_1+j_2}{n_i}$, our sum becomes $\left(\sum_{j|n_i} j\right)/n_i$. Since $\sum_{j|n_i} j = 2n_i$, we have $\sum_{j|n_i} 1/j = 2$ as desired.

11. Let O be a circle with chord AB. The perpendicular bisector to AB is drawn, intersecting O at points C and D, and intersecting AB at the midpoint E. Finally, a circle O' with diameter ED is drawn, and intersects the chord AD at the point F. Given $EC = 12$, and $EF = 7$, compute the radius of O.

Solution: Let $\theta = \angle CDA$ and let R be the radius of O. Note that $\angle CAD$, $\angle AED$, and $\angle EFD$ are right angles, and $\angle ADE = \angle EDF = \theta$, so $\frac{AD}{CD} = \frac{ED}{AD} = \cos\theta$, while $\frac{EF}{ED} = \sin\theta$. Then since $CD = 2R$, we have $ED = 2R\cos^2\theta$, hence $EF = 2R\cos^2\theta\sin\theta$ and $EC = CD - ED = 2R\sin^2\theta$. This gives

$$2R \cdot EF^2 = EC \cdot (2R - EC)^2,$$

i.e. $98R = 48(R-6)^2$, which we can rewrite as

$$0 = 24(R-6)^2 - 49R = (8R-27)(3R-32),$$

so the only possible values of R are $\frac{27}{8}$ and $\frac{32}{3}$. Since we must have $2R = CD > EC = 12$, R cannot be $\frac{27}{8}$, so we must have $R = \boxed{32/3}$.

12. Suppose r, s, t are the roots of the polynomial $x^3 - 2x + 3$. Find

$$\frac{1}{r^3-2} + \frac{1}{s^3-2} + \frac{1}{t^3-2}$$

Solution: Observe that

$$\frac{1}{r^3-2} + \frac{1}{s^3-2} + \frac{1}{t^3-2} = \frac{1}{2r-5} + \frac{1}{2s-5} + \frac{1}{2t-5}$$

since $r^3 - 2 = 2r - 5$ if and only if $r^3 - 2r + 3 = 0$. Therefore our answer is

$$\frac{4(rs+st+tr) - 20(r+s+t) + 75}{8rst - 20(rs+st+tr) + 50(r+s+t) - 125}$$

which equals $\frac{-8+75}{-24+20\cdot 2-125} = \boxed{-67/109}$ by Vieta.

13. Let $a_1, a_2, \ldots, a_{14}$ be points chosen independently at random from the interval $[0,1]$. For $k = 1, 2, \ldots, 7$, let I_k be the closed interval lying between a_{2k-1} and a_{2k} (from the smaller to the larger). What is the probability that the intersection of $I_1, I_2, \ldots, I_7$ is nonempty?

Solution: With probability 1, all a_i are distinct, and the order of the a_i is uniform over all $14!$ permutations of $\{1, 2, \ldots, 14\}$, meaning that for each permutation σ, the probability that $a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(14)}$ is $\frac{1}{14!}$. Now, $I_1, I_2, \ldots, I_7$ intersect exactly when there is some $c \in [0,1]$ such that for $k = 1, \ldots, 7$, either $a_{2k-1} < c < a_{2k}$ or $a_{2k} < c < a_{2k-1}$. Such a c exists iff for each k exactly one of a_{2k-1}, a_{2k} is among the first 7 of the a_i, that is, for each k exactly one of $2k-1, 2k$ is in S where S is the set of indices of the smallest 7 a_i. But S is uniformly distributed over all $\binom{14}{7}$ subsets of $\{1, 2, \ldots, 14\}$ of size 7, while there are 2^7 choices of S containing exactly one of $2k-1, 2k$ for $k = 1, \ldots, 7$ (given by choosing, for each k, either $2k-1$ or $2k$ to belong to S). Thus the desired probability is $\frac{2^7}{\binom{14}{7}} = \boxed{16/429}$.

14. Consider all triangles $\triangle ABC$ with area $144\sqrt{3}$ such that

$$\frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} = \frac{1}{4}$$

Over all such triangles ABC, what is the smallest possible perimeter?

Solution: Let R be the circumradius, and r the inradius. By the Law of Sines, we have that

$$\frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} = \frac{1}{4} \implies \frac{8R^3 \sin A \sin B \sin C}{2R(\sin A + \sin B + \sin C)} = \frac{abc}{a+b+c} = R^2$$

However, $[ABC] = \frac{abc}{4R} = \frac{r(a+b+c)}{2} \implies \frac{abc}{a+b+c} = 2rR = R^2 \implies R = 2r$. By Euler's inequality, ABC is equilateral, and has unique side length. Call the side length s. Then since $\frac{s^2\sqrt{3}}{4} = 144\sqrt{3}$, we have $s = \sqrt{576} = 24$, which gives a perimeter of $\boxed{72}$.

15. Let N be the number of sequences $(x_1, x_2, \ldots, x_{2018})$ of elements of $\{1, 2, \ldots, 2019\}$, not necessarily distinct, such that $x_1 + x_2 + \cdots + x_{2018}$ is divisible by 2018. Find the last three digits of N.

Solution: For convenience, let $S_n = \{1, 2, \ldots, n+1\}$. Let $f_0(n)$ be the number of ways to select n elements from S_n so that the sum of the elements is divisible by n. Furthermore, in general define $f_i(n)$ to be the number of ways to select $n-i$ elements from s_n so that the sum of the elements is $-i \pmod{n}$, for all $0 \le i \le n-1$. Then we have the recurrence

$$f_0(n) = 1 \cdot ((n+1)^{n-1} - f_1(n)) + 2 \cdot f_1(n) = (n+1)^{n-1} + f_1(n)$$

because if we consider choosing the first $n-1$ elements of the sequence, if the sum of the $n-1$ elements we choose is not $-1 \pmod{n}$, we have exactly one choice for our nth element to make the sum $0 \pmod{n}$, and if the sum is $-1 \pmod{n}$, we have two ways, namely 1 and $n+1$.

Similarly, we have the recursion for general i,

$$f_i(n) = (n+1)^{n-i-1} + f_{i+1}(n)$$

Summing over all i, we use

$$f_0(n) = (n+1)^{n-1} + f_1(n)$$
$$f_1(n) = (n+1)^{n-2} + f_2(n)$$
$$\cdots$$
$$f_{n-2}(n) = (n+1)^1 + f_{n-1}(n)$$

and the fact that $f_{n-1}(n) = 2$, since we can choose 1 or $n+1$, to get that

$$f_0(n) = 2 + (n+1) \cdot \frac{(n+1)^{n-1} - 1}{n}$$

Hence, we just wish to find $f_0(2018) = 2 + 2019 \cdot \frac{2019^{2017}-1}{2018}$ mod 1000. To find this mod 8, since $2019^2 \equiv 3^2 \equiv 1 \pmod{8}$ we see that

$$2 + 2019 \cdot (2019^{2016} + 2019^{2015} + \cdots + 2019 + 1) \equiv 2 + 3 \cdot (1) \equiv 5 \pmod{8}$$

To find this modulo 125: after a little work, we have that the inverse of 2018 mod 125 is 7 and that $19^{17} \equiv 64$ mod 125, hence by Euler's theorem,

$$2 + 2019 \cdot \frac{2019^{2017} - 1}{2018} \equiv 2 + 19 \cdot 7 \cdot (19^{17} - 1) \equiv 8 \cdot 19^{17} - 6 \equiv 8 \cdot 64 - 6 \equiv 6 \pmod{125}$$

Thus the answer is $\boxed{381}$.