Individual Round

Problem 1. Consider a cube with side length 2. Take any one of its vertices and consider the three midpoints of the three edges emanating from that vertex. What is the distance from that vertex to the plane formed by those three midpoints?

Problem 2. Digits H, M, and C satisfy the following relations where $\overline{ABC}$ denotes the number whose digits in base 10 are A, B, and C.

$$\overline{H} \times \overline{H} = \overline{M} \times \overline{C} + 1$$
$$\overline{HH} \times \overline{H} = \overline{MC} \times \overline{C} + 1$$
$$\overline{HHH} \times \overline{H} = \overline{MCC} \times \overline{C} + 1$$

Find $\overline{HMC}$.

Problem 3. Two players play the following game on a table with fair two-sided coins. The first player starts with one, two, or three coins on the table, each with equal probability. On each turn, the player flips all the coins on the table and counts how many coins land heads up. If this number is odd, a coin is removed from the table. If this number is even, a coin is added to the table. A player wins when he/she removes the last coin on the table. Suppose the game ends. What is the probability that the first player wins?

Problem 4. Cyclic quadrilateral $[BLUE]$ has right $\angle E$. Let R be a point not in $[BLUE]$. If $[BLUR] = [BLUE]$, $\angle ELB = 45°$, and $\overline{EU} = \overline{UR}$, find $\angle RUE$.

Problem 5. There are two tracks in the x, y plane, defined by the equations

$$y = \sqrt{3 - x^2} \qquad \text{and} \qquad y = \sqrt{4 - x^2}$$

A baton of length 1 has one end attached to each track and is allowed to move freely, but no end may be picked up or go past the end of either track. What is the maximum area the baton can sweep out?

Problem 6. For integers $1 \leq a \leq 2$, $1 \leq b \leq 10$, $1 \leq c \leq 12$, $1 \leq d \leq 18$, let $f(a, b, c, d)$ be the unique integer between 0 and 8150 inclusive that leaves a remainder of a when divided by 3, a remainder of b when divided by 11, a remainder of c when divided by 13, and a remainder of d when divided by 19. Compute

$$\sum_{a+b+c+d=23} f(a, b, c, d).$$

Problem 7. Compute $\cos(\theta)$ if $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{3^n} = 1$.

Problem 8. How many solutions does this equation $\left(\frac{a+b}{2}\right)^2 = \left(\frac{b+c}{2019}\right)^2$ have in positive integers a, b, c that are all less than 2019^2?

Problem 9. Consider a square grid with vertices labeled $1, 2, 3, 4$ clockwise in that order. Fred the frog is jumping between vertices, with the following rules: he starts at the vertex label 1, and at any given vertex he jumps to the vertex diagonally across from him with probability $\frac{1}{2}$ and the vertices adjacent to him each with probability $\frac{1}{4}$. After 2019 jumps, suppose the probability that the sum of the labels on the last two vertices he has visited is 3 can be written as $2^{-m} - 2^{-n}$ for positive integers m, n. Find $m + n$.

Problem 10. The base ten numeral system uses digits 0-9 and each place value corresponds to a power of 10. For example,

$$2019 = 2 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 9 \cdot 10^0.$$

Let $\phi = \dfrac{1+\sqrt{5}}{2}$. We can define a similar numeral system, base ϕ, where we only use digits 0 and 1, and each place value corresponds to a power of ϕ. For example,

$$11.01 = 1 \cdot \phi^1 + 1 \cdot \phi^0 + 0 \cdot \phi^{-1} + 1 \cdot \phi^{-2}.$$

Note that base ϕ representations are not unique, because, for example, $100_\phi = 11_\phi$. Compute the base ϕ representation of 7 with the fewest number of 1s.

Problem 11. Let ABC be a triangle with $\angle BAC = 60$ and with circumradius 1. Let G be its centroid and D be the foot of the perpendicular from A to BC. Suppose $AG = \frac{\sqrt{6}}{3}$. Find AD.

Problem 12. Let $f(a, b)$ be a function with the following properties for all positive integers $a \neq b$:

$$f(1,2) = f(2,1)$$
$$f(a,b) + f(b,a) = 0$$
$$f(a+b,b) = f(b,a) + b$$

Compute:

$$\sum_{i=1}^{2019} f(4^i - 1, 2^i) + f(4^i + 1, 2^i)$$

Problem 13. You and your friends have been tasked with building a cardboard castle in the two-dimensional Cartesian plane. The castle is built by the following rules:

1. There is a tower of height 2^n at the origin.
2. From towers of height $2^i \geq 2$, a wall of length 2^{i-1} can be constructed between the aforementioned tower and a new tower of height 2^{i-1}. Walls must be parallel to a coordinate axis, and each tower must be connected to at least one other tower by a wall.

If one unit of tower height costs \$9 and one unit of wall length costs \$3 and $n = 1000$, how many distinct costs are there of castles that satisfy the above constraints? Two castles are distinct if there exists a tower or wall that is in one castle but not in the other.

Problem 14. For n digits, $(a_1, a_2, \ldots, a_n)$ with $0 \le a_i < n$ for $i = 1, 2, \ldots, n$ and $a_1 \ne 0$ define $(\overline{a_1 a_2 \ldots a_n})_n$ to be the number with digits $a_1, a_2, \ldots a_n$ written in base n.

Let $S_n = \{(a_1, a_2, a_3, ..., a_n) | \ (n+1) | \ (\overline{a_1 a_2 a_3 ... a_n})_n, \ a_1 \ge 1\}$ be the set of n-tuples such that $(\overline{a_1 a_2 \ldots a_n})_n$ is divisible by $n + 1$.

Find all $n > 1$ such that n divides $|S_n| + 2019$.

Problem 15. Let $\mathcal{P}$ be the set of polynomials with degree 2019 with leading coefficient 1 and non-leading coefficients from the set $\mathcal{C} = \{-1, 0, 1\}$. For example, the function $f = x^{2019} - x^{42} + 1$ is in $\mathcal{P}$, but the functions $f = x^{2020}$, $f = -x^{2019}$, and $f = x^{2019} + 2x^{21}$ are not in $\mathcal{P}$.

Define a *swap* on a polynomial f to be changing a term ax^n to bx^n where $b \in \mathcal{C}$ and there are no terms with degree smaller than n with coefficients equal to a or b. For example, a swap from $x^{2019} + x^{17} - x^{15} + x^{10}$ to $x^{2019} + x^{17} - x^{15} - x^{10}$ would be valid, but the following swaps would not be valid:

$$\begin{array}{rcc} x^{2019} + x^3 & \text{to} & x^{2019} \\ x^{2019} + x^3 & \text{to} & x^{2019} + x^3 + x^2 \\ x^{2019} + x^2 + x + 1 & \text{to} & x^{2019} - x^2 - x - 1 \end{array}$$

Let $\mathcal{B}$ be the set of polynomials in $\mathcal{P}$ where all non-leading terms have the same coefficient. There are p polynomials that can be reached from each element of $\mathcal{B}$ in exactly s *swaps*, and there exist 0 polynomials that can be reached from each element of $\mathcal{B}$ in less than s *swaps*.

Compute $p \cdot s$, expressing your answer as a prime factorization.