Team Round

Problem 1. Let ABC be an equilateral triangle of side length 6. Points D, E and F are on sides AB, BC, and AC respectively such that $AD = BE = CF = 2$. Let circle O be the circumcircle of DEF, i.e. the circle that passes through points D, E, and F. What is the area of the region inside triangle ABC but outside circle O?

Solution. $\boxed{6\sqrt{3} - 2\pi}$

We will use complementary counting. The area of ABC is $\frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$. The region inside both the triangle and the circle consist of 3 triangles and 3 sectors of circle O. Each of the triangles has a base of length 2 on one of the side lengths of ABC and a height of $\sqrt{3}$, so they each have area $\sqrt{3}$, and together have an area of $3\sqrt{3}$. The triangles each cut out 60 degrees from circle O, meaning the three sectors that are in both the triangle and the circle together form a 180 degree sector of O. The radius of circle O is the hypotenuse of the small triangles, so circle O has radius 2 and the sectors together have area $4^2 \cdot \pi \cdot \frac{1}{2} = 8\pi$. Thus, the area inside both circle O and triangle ABC is $3\sqrt{3} + 2\pi$, so subtracting from the area of ABC, we find that the area inside triangle ABC but outside circle O is $6\sqrt{3} - 2\pi$.

Problem 2. Alex, Bob, Charlie, Daniel, and Ethan are five classmates. Some pairs of them are friends. How many possible ways are there for them to be friends such that everyone has at least one friend, and such that there is exactly one loop of friends among the five classmates?

Note: friendship is two-way, so if person x is friends with person y then person y is friends with person x. Casework on max length of the loop, the possible cases are $3, 4, 5$.

Solution. $\boxed{232}$

1. If it is 5: adding any more edges causes more than one loop, hence we have $5!/5 \cdot \frac{1}{2} = 12$ possibilities in this case.
2. If it is 4: we can choose the loop of 4 people in exactly $\binom{5}{4} \cdot 4!/4 \cdot \frac{1}{2} = 15$ ways. Adding any edge between two vertices in the loop creates another loop. Let v be the vertex in the graph not in the loop. Since v must be connected to another vertex but as the graph no more loops, v must be connected to 1 other vertex, and there are 4 choices for this vertex. In total there are $15 \cdot 4 = 60$ possibilities in this case.
3. If it is 3: we can choose the loop of 3 people in exactly $\binom{5}{3} \cdot 1 = 10$ ways. Let the two vertices in the graph not in the loop be v and u. If v and u are not connected, then note each of v, u must be connected to exactly one vertex in the loop, since the graph is connected. This gives $3^2 = 9$ possibilities. If v and u are connected, we do more cases. If u, v are connected to each other, we have 1 possibility. Both u, v cannot be both connected to a vertex in the loop. If v is not connected to a vertex in the loop but u is, this gives 3 possibilities. u not being connected to a vertex in the loop but v being connected is the same case. In total there are $10 \cdot (9 + 3 + 3 + 1) = 160$ possibilities in this case.

In total the answer is $12 + 60 + 160 = 232$.

Problem 3. A frog is jumping between lattice points on the coordinate plane in the following way: On each jump, the frog randomly goes to one of the 8 closest lattice points to it, such that the frog never goes in the same direction on consecutive jumps. If the frog starts at $(20, 19)$ and jumps to $(20, 20)$, then what is the expected value of the frog's position after it jumps for an infinitely long time?

Solution. $\boxed{(20, \frac{159}{8})}$
Shift the coordinate system such that the frog is at the origin. Now, the expected value of the frog's final position if the frog's second jump in a direction θ is the opposite of the expected value if the frog's second jump is in the direction $-\theta$. Thus, all the expected values cancel out except for the expected value if the frog moves down. Since the frog moves down with a probability $\frac{1}{7}$, the expected value of the frog's final position is $\frac{1}{7} \cdot E$the expected value if the frog moves to (0,-1)]. If the expected value of the frog's final position is $(0, x)$, then $x = \frac{1}{7}(-1 - x)$, so $7x = -1 - x \to 8x = -1 \to x = -\frac{1}{8}$. Shifting the coordinates back, the frog's expected position as it infinitely jumps is $(20, 20 - \frac{1}{8}) = (20, \frac{159}{8})$.

Problem 4. Let ΔABC be a triangle such that the area $[ABC] = 10$ and $\tan(\angle ABC) = 5$. If the smallest possible value of $\left(\overline{AC}\right)^2$ can be expressed as $-a + b\sqrt{c}$ for positive integers a, b, c, what is $a + b + c$?

Solution. $\boxed{42}$
Let $t = \tan B, K = [ABC]$. Then $\frac{1}{2}ac\sin B = K \implies ac\cos B = \frac{2K}{t} \implies b^2 = a^2 + c^2 - 2ac\cos B = a^2 + c^2 - \frac{4K}{t} \geq 2ac - \frac{4K}{t}$. So the answer is minimized when $a = c$. Then compute $\sin^2 \frac{B}{2} = \frac{\sqrt{26}-1}{2\sqrt{26}}$ by half angle, so $\frac{1}{2}a^2 \sin B = 10 \implies a^2 = 4\sqrt{26} \implies b^2 = (2a\sin\frac{B}{2})^2 = 8\sqrt{26} - 8$.

Problem 5. A tournament has 5 players and is in round-robin format (each player plays each other exactly once). Each game has a $\frac{1}{3}$ chance of player A winning, a $\frac{1}{3}$ chance of player B winning, and a $\frac{1}{3}$ chance of ending in a draw. The probability that at least one player draws all of their games can be written in simplest form as $\frac{m}{3^n}$ where m, n are positive integers. Find $m + n$.

Solution. $\boxed{3411}$
We use the principle of inclusion-exclusion. The probability of one person drawing all of their matches is $\frac{1}{3^5}$. In general we want to know the probability of some k players drawing all their games. There are $\binom{5}{k}$ ways to choose the drawing players. Collectively these players participate in $\binom{5}{2} - \binom{k-1}{2} = 10 - \binom{k-1}{2}$ games. Hence, the probability that k players draw all their matches is $\frac{\binom{5}{k}}{3^{10-\binom{k-1}{2}}}$. Therefore, by inclusion-exclusion, the probability we are looking for is

$$\begin{aligned}
&\frac{\binom{5}{1}}{3^4} - \frac{\binom{5}{2}}{3^7} + \frac{\binom{5}{3}}{3^9} - \frac{\binom{5}{4}}{3^{10}} + \frac{\binom{5}{5}}{3^{10}} \\
&= \frac{1}{3^{10}}\left(5 \cdot 3^6 - 10 \cdot 3^3 + 10 \cdot 3 - 5 + 1\right) \\
&= \frac{1}{3^{10}}\left(5 \cdot 729 - 270 + 30 - 4)\right) \\
&= \frac{1}{3^{10}}\left(3645 - 244)\right) \\
&= \frac{3401}{3^{10}}
\end{aligned}$$

From which we get $3401 + 10 = \boxed{3411}$.

Problem 6. Compute $\prod_{i=1}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1)$.

Solution. $\boxed{\frac{2^{2^{2020}}+2^{2^{2019}}+1}{7}}$
Multiplying the product by $7 = (2^2+2+1)$ and repeatedly using the identity $(x^2-x+1)(x^2+x+1) = x^4+x^2+1$, we have

$$\begin{aligned}(2^2+2+1)\prod_{i=1}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1) &= (2^2+2+1)(2^2-2+1)\prod_{i=2}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1)\\ &= (2^4+2^2+1)\prod_{i=2}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1)\\ &= (2^4+2^2+1)(2^4-2^2+1)\prod_{i=3}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1)\\ &= \cdots\\ &= (2^{2^{2019}}+2^{2^{2018}}+1)(2^{2^{2019}}-2^{2^{2018}}+1)\\ &= 2^{2^{2020}}+2^{2^{2019}}+1\end{aligned}$$

thus yielding the answer of $\frac{2^{2^{2020}}+2^{2^{2019}}+1}{7}$.

Problem 7. Let S be the set of all positive integers n satisfying the following two conditions:

- n is relatively prime to all positive integers less than or equal to $\frac{n}{6}$.
- $2^n \equiv 4 \mod n$

What is the sum of all numbers in S?

Solution. $\boxed{16}$, $\boxed{23}$
By Fermat's Little Theorem, $a^p \equiv a \mod p$ so no primes greater than 2 are in S. If $n > 100$ then the first condition means that n is relatively prime to all integers less than $\sqrt{n}$, so n is prime, meaning that all elements of S are less than 100. Now, we can list all the numbers less than 100 that satisfy the first condition and are not prime. This yields a fairly small set of potential elements of S: 1,2,4,6,8,9,10,15,25. We can calculate $2^n \mod n$ for each of these values by calculating $2^{n-\phi(n)} \mod n$, which is a relatively simple calculation, and we find that the only numbers that are elements of S are 1, 2, 4, 6,and 10, so the sum of these elements is 23.

Problem 8. Consider an infinite sequence of reals $x_1, x_2, x_3, \ldots$ such that $x_1 = 1, x_2 = \frac{2\sqrt{3}}{3}$ and with the recursive relationship

$$n^2(x_n - x_{n-1} - x_{n-2}) - n(3x_n + 2x_{n-1} + x_{n-2}) + (x_n x_{n-1} x_{n-2} + 2x_n) = 0$$

Find x_{2019}. [1]

[1] This problem was thrown out due to an error in the problem statement. The correct recursive relationship should have been: $n^2(x_n - x_{n-1} - x_{n-2}) - n(3x_n - 2x_{n-1} - x_{n-2}) + (2x_n - x_n x_{n-1} x_{n-2}) = 0$.

Solution. $\boxed{2019\sqrt{3}+4038}$
The key insight is to notice that the recursive relationship rewrites as, upon solving for x_n in terms of constants and other variables,

$$x_n = n \cdot \frac{\frac{x_{n-1}}{n-1} + \frac{x_{n-2}}{n-2}}{1 - \frac{x_{n-1}x_{n-2}}{(n-1)(n-2)}}$$

Thus $x_1/1, x_2/2, x_3/3, \ldots$ forms a sequence given by $\arctan(\frac{x_n}{n}) = \arctan(\frac{x_{n-1}}{n-1}) + \arctan(\frac{x_{n-2}}{n-2})$. Looking at initial terms of the sequence, we see that $\frac{1}{\pi}\arctan(\frac{x_n}{n})$ forms the following pattern with period 24:

$$1/4, 1/6, 5/12, -5/12, 0, -5/12, -5/12, 1/6, -1/4, -1/12, -1/3, -5/12,$$

$$1/4, -1/6, 1/12, -1/12, 0, -1/12, -1/12, -1/6, -1/4, -5/12, 1/3, -1/12$$

Thus $x_{2019} = 2019 \cdot \tan(\frac{5\pi}{12}) = 2019(\sqrt{3}+2) = 2019\sqrt{3} + 4038$.

Problem 9. Consider a rectangle with length 6 and height 4. A rectangle with length 3 and height 1 is placed inside the larger rectangle such that it is distance 1 from the bottom and leftmost sides of the larger rectangle.

We randomly select one point from each side of the larger rectangle, and connect these 4 points to form a quadrilateral. What is the probability that the smaller rectangle is strictly contained within that quadrilateral?

Solution. $\boxed{\frac{9-\ln 3-3\ln 2}{18}}$
The problem is equivalent to the following: randomly select one point from the bottom side and one point from the top side of the larger rectangle. Draw two lines connecting the point on the bottom side to the two lower vertices of the smaller rectangle, and two lines connecting the point on the top side to the two upper vertices of the smaller rectangle. Consider the two intersections of the lines connecting the top and bottom point to the two left points of the smaller rectangle with the left side of the larger rectangle, and look at the region above the intersection with the upper line and below the lower line. Consider the same process for the right side of the larger rectangle. What is the probability that both of these regions are nonempty?

To this end place the bottom left corner of the larger rectangle at the origin of the Cartesian plane and put the bottom side of the larger rectangle as the positive x axis and the left side of the larger rectangle as the positive y axis. Suppose the bottom point is distance a from the rectangle's left side and suppose the top point is distance b from the rectangle's left side. Then the two lines intersecting the left side of the larger rectangle are $y = \frac{-1}{a-1}x + \frac{a}{a-1}, y = \frac{2}{b-1}x + \frac{2b-4}{b-1}$ going through the bottom and top points respectively, and the two lines intersecting the right side of the larger rectangle are $y = \frac{1}{4-a}x - \frac{a}{4-a}, y = \frac{-2}{4-b}x + \frac{16-2b}{4-b}$ going through the bottom and top points respectively. From here we can find the intersections on the left side to be $\frac{a}{a-1}$, $\frac{2b-4}{b-1}$ and $\frac{6-a}{4-a}$, $\frac{4-2b}{4-b}$ respectively.

So, what we want to find is $P(\frac{a}{a-1} > \frac{2b-4}{b-1}) \leftrightarrow P(\frac{1}{a-1} > \frac{b-3}{b-1}) \leftrightarrow P(\frac{(2b+3a)-(ab+4)}{(a-1)(b-1)} > 0)$ and that $P(\frac{6-a}{4-a} > \frac{4-2b}{4-b}) \leftrightarrow P(\frac{2}{4-a} > \frac{-b}{4-b}) \leftrightarrow P(\frac{8+2b-ab}{(4-a)(4-b)} > 0)$. Since a, b are chosen uniformly between 0 and 6, upon graphing we find that this is equivalent to finding the area of the plane with a as the x axis and $0 \le a \le 6$ and b as the y axis with $0 \le b \le 6$, with the conditions that $1 < a < 4, 1 < b < 4$, or $a > 4$ and $\frac{3a-4}{a-2} > b > \frac{8}{a-2}$, or $a < 1$ and $\frac{3a-4}{a-2} > b > 1$, or $a < \frac{3a-4}{a-2}$ and $b > 1$.

We compute this area to be

$$3^2 + \left(\int_0^{4/3} \frac{3x-4}{x-2} dx - 1 \right) + \int_4^6 \frac{3x-4-8}{x-2} dx = 9 + (-1 + 4 + 2\ln\frac{1}{3}) + 3(2 - 2\ln 2)$$

So our final answer is $\frac{18-2\ln 3-6\ln 2}{36} = \frac{9-\ln 3-3\ln 2}{18}$.

Problem 10. n players are playing a game. Each player has n tokens. Every turn, two players with at least one token are randomly selected. The player with less tokens gives one token to the player with more tokens. If both players have the same number of tokens, a coin flip decides which player receives a token and which player gives a token. The game ends when one player has all the tokens. If $n = 2019$, suppose the maximum number of turns the game could take to end can be written as $\frac{1}{d}(a \cdot 2019^3 + b \cdot 2019^2 + c \cdot 2019)$ for integers a, b, c, d. Find $\frac{abc}{d}$.

Solution. $\boxed{-\frac{11}{2}}$

We claim that the answer is $\frac{(11n-1)(n)(n-1)}{24}$ for odd n.

Let t_i be the number of tokens of the player with the ith lowest number of tokens. We first define the *entropy* of the game state as $S = \sum_{i=1}^{n} (t_i - n)^2$. We note that for a move between players i and j, S always increases, since:

$$(t_i - 1 - n)^2 + (t_j + 1 - n)^2 = (t_i - n)^2 + (t_j - n)^2 + 2 \cdot (t_j - t_i) + 2$$

This means that S increases by $2 \cdot (t_j - t_i) + 2$, which is always at least 2 since $t_j \geq t_i$.

Let a game state be *monotonous* if for all $i \neq j$, if $0 < t_i, t_j$, then $t_i \neq t_j$. Let C be the *monotonous* state with minimum S of S_C. We note that all *monotonous* states are reachable from C.

C can be reached in at most $\frac{S_C}{2}$ turns. This is because the initial state has $S = 0$ and C has $S = S_C$, and the minimum entropy increase per turn is 2.
For odd n:

$$\frac{S_C}{2} = \frac{2 \cdot \sum_{i=1}^{\frac{n-1}{2}} i^2}{2} = \frac{\frac{n-1}{2} \cdot \frac{n+1}{2} \cdot n}{6} = \frac{(n+1)(n)(n-1)}{24}$$

Once we obtain C, we note that the t_i tokens of player i can go through at most $n - i$ turns per token before reaching player n. We can now calculate the maximum number of turns to game completion from C, which is $\sum_{i=1}^{n} t_i \cdot (n - i)$.
For odd n:

$$\sum_{i=1}^{n} \left(i + \frac{n-1}{2} \right) \cdot (n - i) = \left(\frac{n+1}{2} \sum_{i=1}^{n} i \right) - \left(\sum_{i=1}^{n} i^2 \right) + \left(\frac{n-1}{2} \cdot n \sum_{i=1}^{n} 1 \right)$$

$$= \frac{n(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} + \frac{n^2(n-1)}{2} = \frac{(10n-2)(n)(n-1)}{24}$$

Finally, we compute the total number of turns to game completion.
For odd n:

$$\frac{(10n-2)(n)(n-1)}{24} + \frac{(n+1)(n)(n-1)}{24} = \frac{(11n-1)(n)(n-1)}{24}$$

Plugging in $n = 2019$, we have:

$$\frac{(11 \cdot 2019 - 1)(2019)(2019 - 1)}{24} = \frac{1}{24} \cdot (11 \cdot 2019^3 - 12 \cdot 2019^2 + 1 \cdot 2019)$$

Hence $d = 24, a = 11, b = -12, c = 1$. So:

$$\frac{abc}{d} = \frac{11 \cdot (-12) \cdot 1}{24} = -\frac{11}{2}$$

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Individual Round

Problem 1. Consider a cube with side length 2. Take any one of its vertices and consider the three midpoints of the three edges emanating from that vertex. What is the distance from that vertex to the plane formed by those three midpoints?

Solution. $\boxed{\frac{\sqrt{3}}{3}}$
Consider the tetrahedron formed by the three midpoints and the given vertex. Calculate the volume of this tetrahedron in two ways. Letting the answer be x we have $\frac{1}{6} \cdot 1^3 = \frac{1}{3} \cdot x \cdot \frac{\sqrt{2}^2\sqrt{3}}{4} \implies x = \frac{\sqrt{3}}{3}$.

Problem 2. Digits H, M, and C satisfy the following relations where $\overline{ABC}$ denotes the number whose digits in base 10 are A, B, and C.

$$\begin{aligned} \overline{H} \times \overline{H} &= \overline{M} \times \overline{C} + 1 \\ \overline{HH} \times \overline{H} &= \overline{MC} \times \overline{C} + 1 \\ \overline{HHH} \times \overline{H} &= \overline{MCC} \times \overline{C} + 1 \end{aligned}$$

Find $\overline{HMC}$.

Solution. $\boxed{435}$
We have

$$\begin{aligned} H^2 &= MC + 1 \\ 11H^2 &= (10M + C)C + 1 \end{aligned}$$

Subtracting 10 times the first equation from the second, we get

$$\begin{aligned} H^2 \cdot (11 - 10) &= 10MC + C^2 + 1 - 10MC - 10 \\ C^2 - H^2 &= 9 \end{aligned}$$

which has the unique solution $H = 4$, $C = 5$ (corresponding to a 3-4-5 Pythagorean triple) over digits 0-9. From here it is easy to find $M = 3$, hence the answer $\overline{HMC} = 435$.

Problem 3. Two players play the following game on a table with fair two-sided coins. The first player starts with one, two, or three coins on the table, each with equal probability. On each turn, the player flips all the coins on the table and counts how many coins land heads up. If this number is odd, a coin is removed from the table. If this number is even, a coin is added to the table. A player wins when he/she removes the last coin on the table. Suppose the game ends. What is the probability that the first player wins?

Solution. $\boxed{\frac{2}{3}}$
Note the parity of the number of coins when each player is removing a coin is fixed based on the initial number of coins on the table. Since the game ends, the number of coins when the first player removes a coin must be odd before the removal and even after, since 0 is even. Moreover, if this is the case, because we are given that the game ends, then the first player will win. Thus the first player is guaranteed to

win if and only if the number of coins at the start is odd, so the answer is $\frac{2}{3}$.

Problem 4. Cyclic quadrilateral $[BLUE]$ has right $\angle E$. Let R be a point not in $[BLUE]$. If $[BLUR] = [BLUE]$, $\angle ELB = 45°$, and $\overline{EU} = \overline{UR}$, find $\angle RUE$.

Solution. $\boxed{90°}$
$[BLUR] = [BLUE] \implies [BUE] = [BUR]$. Let d_E be the distance from E to BU and d_R be the distance R to BU. Thus $d_E = d_R$, so BU and ER are parallel. Thus $\angle UER = \angle BUE = \angle BLE = 45°$. EUR is isosceles so $\angle RUE$ is $90°$.

Problem 5. There are two tracks in the x, y plane, defined by the equations

$$y = \sqrt{3 - x^2} \qquad \text{and} \qquad y = \sqrt{4 - x^2}$$

A baton of length 1 has one end attached to each track and is allowed to move freely, but no end may be picked up or go past the end of either track. What is the maximum area the baton can sweep out?

Solution. $\boxed{\frac{5\pi}{12}}$
Check that the tracks are semicircles in the positive y plane. Note the baton must be always tangent to the inner track since the baton and the radii of the two semicircular tracks from a $1 - \sqrt{3} - 2$ right triangle. We cannot rotate the baton, so we the point of the baton on the inner track must go from $0°$ to $150°$ on the inner track, and due to the tangency, from the inner point of the baton the outer point of the baton has at most 2 valid positions. Thus the answer is $(2^2 - \sqrt{3}^2)\pi - (\frac{\sqrt{3}}{2} - \frac{3\pi}{12}) - (\frac{4\pi}{12} - \frac{\sqrt{3}}{2}) = \frac{5\pi}{12}$.

Problem 6. For integers $1 \le a \le 2$, $1 \le b \le 10$, $1 \le c \le 12$, $1 \le d \le 18$, let $f(a, b, c, d)$ be the unique integer between 0 and 8150 inclusive that leaves a remainder of a when divided by 3, a remainder of b when divided by 11, a remainder of c when divided by 13, and a remainder of d when divided by 19. Compute

$$\sum_{a+b+c+d=23} f(a, b, c, d).$$

Solution. $\boxed{945516}$
Note if $a + b + c + d = 23$ then $(3 - a) + (11 - b) + (13 - c) + (19 - d) = 46 - (a + b + c + d) = 23$ and that $f(a, b, c, d) = 8151 - f(3 - a, 11 - b, 13 - c, 19 - d)$ as $a, b, c, d \neq 0$. Hence the answer is half the number of nonzero residues a, b, c, d modulo $3, 11, 13, 19$ respectively that sum to 23 times 8151.

To count the number of such a, b, c, d we do casework.

- $11 \le d \le 18$: we have $a + b + c = 23 - d$. Doing cases on c, $2 \le a + b \le 22 - d$ which yields $1, 2, \ldots, 2$ solutions for a, b in each case as $1 \le a \le 2$. Thus there are $-2d + 41$ solutions for each case.
- $9 \le d \le 10$: we have $a + b + c = 23 - d$. Doing cases on c, $2 \le a + b \le 12$ which yields $1, 2, \ldots, 2, 1$ solutions for a, b in each case. Thus there are $9 \cdot 2 + 2 = 20$ solutions for each case.
- $1 \le d \le 8$: we have $a + b + c = 23 - d$. Doing cases on c, $-d + 11 \le a + b \le 12$ which yields

$2, \ldots, 2, 1$ solutions for a, b in each case. Thus there are $2d+3$ solutions in each case.

So the number of such a, b, c, d is $2(5+7+\cdots+19+20) = 2(10^2-4)+40 = 232$, hence our answer is $\frac{232}{2} \cdot 8151 = 944516$.

Problem 7. Compute $\cos(\theta)$ if $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{3^n} = 1$.

Solution. $\boxed{\frac{1}{3}}$

We have $\cos(n\theta) = \frac{e^{i(n\theta)}+e^{-i(n\theta)}}{2}$ so

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{3^n} &= \sum_{n=0}^{\infty} \left(\frac{e^{i(n\theta)}}{2 \cdot 3^n} + \frac{e^{-i(n\theta)}}{2 \cdot 3^n} \right) \\ &= \frac{1}{2} \left(\frac{1}{1-\frac{e^{i\theta}}{3}} + \frac{1}{1-\frac{e^{-i\theta}}{3}} \right) \\ &= \frac{1}{2} \left(\frac{2-\frac{2\cos\theta}{3}}{1+\frac{1}{9}-\frac{1}{3}(e^{i\theta}+e^{-i\theta})} \right) \\ &= \frac{9-3\cos\theta}{10-6\cos\theta}\end{aligned}$$

Thus $9-3\cos\theta = 10-6\cos\theta \implies \cos\theta = \frac{1}{3}$.

Problem 8. How many solutions does this equation $\left(\frac{a+b}{2}\right)^2 = \left(\frac{b+c}{2019}\right)^2$ have in positive integers a, b, c that are all less than 2019^2?

Solution. $\boxed{4078384}$

By CRT since $2019 = 3 \cdot 673$ both of which are prime, we can prove that all solutions are given by $a = 2m-b, b, c = 2019m-b$ for some positive integer m.

If $m \leq 2019$ then there are $2m-1$ choices for b for fixed m, hence $\sum_{i=1}^{2019}(2i-1) = 2019^2$ solutions.

If $m > 2019$, note we must have $2019m - b < 2019^2 \implies 2019(m-2019) < b < 2m$. Testing for when this gives a positive amount of valid cases for m we find that we must have $m = 2020, 2021$. $m = 2020$ gives $2020, \ldots, 4039$ as valid values for b, so 2020 valid values for b. $m = 2021$ gives $4039, 4040, 4041$ as valid values for b, so 3 valid values for b.

In total the answer is $2023 + 2019^2 = 4 + 2019 \cdot 2020 = 4078384$ solutions.

Problem 9. Consider a square grid with vertices labeled $1, 2, 3, 4$ clockwise in that order. Fred the frog is jumping between vertices, with the following rules: he starts at the vertex label 1, and at any given vertex he jumps to the vertex diagonally across from him with probability $\frac{1}{2}$ and the vertices adjacent to him each with probability $\frac{1}{4}$. After 2019 jumps, suppose the probability that the sum of the labels on the last two vertices he has visited is 3 can be written as $2^{-m} - 2^{-n}$ for positive integers m, n. Find $m+n$.

Solution. $\boxed{2024}$

Let $p(a,b)$ equal the probability that Fred is at the vertex with label a at time b. Now by induction one can prove that the distribution of states $(p(1,n), p(2,n), p(3,n), p(4,n))$ equals $(\frac{2^{n-1}+1}{2^{n+1}}, \frac{1}{4}, \frac{2^{n-1}-1}{2^{n+1}}, \frac{1}{4})$ for even n and $(\frac{2^{n-1}-1}{2^{n+1}}, \frac{1}{4}, \frac{2^{n-1}+1}{2^{n+1}}, \frac{1}{4})$ for odd n. Seeing this is easier by looking at the transition matrix.

Finally we can compute the answer:

$$p(1,2018)\cdot\frac{1}{4} + p(2,2018)\cdot\frac{1}{4} = \frac{2^{2017}-1}{2^{2021}} + \frac{1}{16} = \frac{2^{2017}+2^{2017}}{2^{2021}} - 2^{2021} = 2^{-3} - 2^{-2021} \implies 2024$$

Problem 10. The base ten numeral system uses digits 0-9 and each place value corresponds to a power of 10. For example,

$$2019 = 2\cdot 10^3 + 0\cdot 10^2 + 1\cdot 10^1 + 9\cdot 10^0.$$

Let $\phi = \dfrac{1+\sqrt{5}}{2}$. We can define a similar numeral system, base ϕ, where we only use digits 0 and 1, and each place value corresponds to a power of ϕ. For example,

$$11.01 = 1\cdot\phi^1 + 1\cdot\phi^0 + 0\cdot\phi^{-1} + 1\cdot\phi^{-2}.$$

Note that base ϕ representations are not unique, because, for example, $100_\phi = 11_\phi$. Compute the base ϕ representation of 7 with the fewest number of 1s.

Solution. $\boxed{10000.0001_\phi}$

The key insight is to notice that $1 = 1_\phi = 0.11_\phi$. In general, we can replace any instance of $\phi^n + \phi^{n+1}$ with ϕ^{n+2} or vice versa.

Using these facts, we see that

$$2 = 1+1 = 1_\phi + .11_\phi = 1.11_\phi = 10.01_\phi = 10.0011_\phi$$

Therefore,

$$\begin{aligned}
4 &= 2+2 = 1.11_\phi + 10.0011_\phi = 11.1111_\phi = 100.1111_\phi\\
5 &= 4+1 = 100.1111_\phi + 1_\phi = 101.1111_\phi = 110.0111_\phi\\
6 &= 5+1 = 110.0111_\phi + 1_\phi = 111.0111_\phi = 1001.0111_\phi = 1001.1001_\phi = 1010.0001_\phi\\
7 &= 6+1 = 1010.0001_\phi + 1_\phi = 1011.0001_\phi = 1100.0001_\phi = 10000.0001_\phi
\end{aligned}$$

It is quite clear that no base ϕ representation of 7 with only one 1 exists, because 7 is not a power of ϕ. Therefore, this is minimal.

Problem 11. Let ABC be a triangle with $\angle BAC = 60$ and with circumradius 1. Let G be its centroid and D be the foot of the perpendicular from A to BC. Suppose $AG = \frac{\sqrt{6}}{3}$. Find AD.

Solution. $\boxed{\frac{3}{4}}$

By Stewart's Theorem we get $AG = \frac{2}{3}\cdot\sqrt{\frac{2b^2+c^2-a^2}{4}} = \frac{\sqrt{2b^2+2c^2-a^2}}{3} \implies 2b^2+2c^2-a^2 = 6$. By equating the area of ABC in two ways we get $\frac{1}{2}a \cdot AD = \frac{1}{2}bc \sin A = \frac{abc}{4R} \implies a = \sqrt{3}, AD = \frac{bc}{2}$. By Law of Cosines we get $b^2 + c^2 - 2bc \cos A = a^2 \implies b^2 + c^2 - bc = 3$. Thus as we have $b^2 + c^2 = \frac{9}{2}$ this means $bc = \frac{3}{2}$ so the answer is $\frac{3}{4}$.

Problem 12. Let $f(a, b)$ be a function with the following properties for all positive integers $a \neq b$:

$$f(1, 2) = f(2, 1)$$
$$f(a, b) + f(b, a) = 0$$
$$f(a + b, b) = f(b, a) + b$$

Compute:

$$\sum_{i=1}^{2019} f(4^i - 1, 2^i) + f(4^i + 1, 2^i)$$

Solution. $\boxed{2^{2021} - 6}$
Just keep applying the formulae until the answer is obtained. The keys here are in realizing that f is anti-commutative, and that $f(a + 2b, b) = f(a, b)$. After some more math we have $f(4^i - 1, 2^i) = 2^{i+1}$ for $i > 1$ and $f(4^i + 1, 2^i) = 0$.

Problem 13. You and your friends have been tasked with building a cardboard castle in the two-dimensional Cartesian plane. The castle is built by the following rules:

1. There is a tower of height 2^n at the origin.
2. From towers of height $2^i \geq 2$, a wall of length 2^{i-1} can be constructed between the aforementioned tower and a new tower of height 2^{i-1}. Walls must be parallel to a coordinate axis, and each tower must be connected to at least one other tower by a wall.

If one unit of tower height costs \$9 and one unit of wall length costs \$3 and $n = 1000$, how many distinct costs are there of castles that satisfy the above constraints? Two castles are distinct if there exists a tower or wall that is in one castle but not in the other.

Solution. $\boxed{4 \cdot 3^{1000} - 5 \cdot 2^{1000} + 1002}$
We claim that the general formula is $4 \cdot 3^n - 5 \cdot 2^n + n + 2$.

Note that the central tower of height 2^n must get built. After this central tower, we can associate to each tower, the wall built to create it. Together, these cost 12·[height of tower], so we are interested in the number of distinct total castle heights (i.e. the sum of heights of each tower in the castle) that can be created.

We claim that for all total castle heights between 2^{n+1} and the max castle height are achievable. This is true because at each branch there are 3 smaller towers with half the cost, so we have enough degrees of freedom to reach every cost. After the central towers, the next smallest towers that can be built are of height $2^{n-1}, 2^{n-2}, \ldots 2, 1$. Therefore, there are $n+1$ castles with heights between 2^n and less than 2^{n+1}.

Therefore the number of distinct castle heights is $n + 2 + M - 2^{n+1}$, where M is the max height of a castle. We can solve for M recursively. Let a_n be the maximum castle height of one of the four branches of a castle (i.e. the max cost of all towers out of its northern tower, for instance). When all possible towers are built we have one castle of height 2^{n-1} and three additional branches of height a_{n-1} each. Thus $a_n = 3a_{n-1} + 2^{n-1}$.

We can solve this recursion by writing, $a_n - 3a_{n-1} = 2^{n-1} = 2 * 2^{n-2} = 2(a_{n-1} - 3a_{n-2})$ and thus $a_n - 5a_{n-1} + 6a_{n-2}$. Using the characteristic polynomial this can be solved as $a_n = u2^n - v3^n$. Plugging in the initial conditions, $a_1 = 1$ and $a_2 = 5$ we get $u = -1$, $v = 1$ and therefore, $a_n = 3^n - 2^n$. Hence $M = 2^n + 4a_n = 4 \cdot 3^n - 3 \cdot 2^n$ and the final answer is $M - 2^{n+1} + n + 2 = 4 \cdot 3^n - 5 \cdot 2^n + n + 2$. Plugging in $n = 1000$ gives the desired answer.

Problem 14. For n digits, $(a_1, a_2, \dots, a_n)$ with $0 \le a_i < n$ for $i = 1, 2, \dots, n$ and $a_1 \neq 0$ define $(\overline{a_1 a_2 \dots a_n})_n$ to be the number with digits $a_1, a_2, \dots a_n$ written in base n.

Let $S_n = \{(a_1, a_2, a_3, ..., a_n) | \ (n+1) | \ (\overline{a_1 a_2 a_3 ... a_n})_n, \ a_1 \ge 1\}$ be the set of n-tuples such that $(\overline{a_1 a_2 \dots a_n})_n$ is divisible by $n + 1$.

Find all $n > 1$ such that n divides $|S_n| + 2019$.

Solution. $\boxed{\{2, 5, 101, 505, 2018\}}$

We know that $|S_n|$ can be approximated by $\lfloor \frac{n^n - n^{n-1} + c}{n+1} \rfloor$ for some constant c.

We then notice that c depends only on if n is even or odd. We consider two cases of n, odd or even.

For n odd, we know that $n^n \equiv -1 \mod (n+1)$ and $n^{n-1} \equiv 1 \mod (n+1)$. Therefore, $|S_n|$ must be equal to $\frac{n^n - n^{n-1} + 2}{n+1} - 1 = \frac{n^n - n^{n-1} - n + 1}{n+1}$ since c must be 2 to make the numerator become $0 \mod (n+1)$. The minus 1 is from taking off the floor function.

Therefore, for n odd, $n | \frac{n^n - n^{n-1} - n + 1}{n+1} + 2019$. which imply that $n | 2019 + 1$ when n is odd.

Similarly for n even, we get $|S_n|$ must be equal to $\frac{n^n - n^{n-1} + n - 1}{n+1}$ and therefore, $n | 2019 - 1$ when n is even.

Problem 15. Let $\mathcal{P}$ be the set of polynomials with degree 2019 with leading coefficient 1 and non-leading coefficients from the set $\mathcal{C} = \{-1, 0, 1\}$. For example, the function $f = x^{2019} - x^{42} + 1$ is in $\mathcal{P}$, but the functions $f = x^{2020}$, $f = -x^{2019}$, and $f = x^{2019} + 2x^{21}$ are not in $\mathcal{P}$.

Define a *swap* on a polynomial f to be changing a term ax^n to bx^n where $b \in \mathcal{C}$ and there are no terms with degree smaller than n with coefficients equal to a or b. For example, a swap from $x^{2019} + x^{17} - x^{15} + x^{10}$ to $x^{2019} + x^{17} - x^{15} - x^{10}$ would be valid, but the following swaps would not be valid:

$$\begin{array}{ccc} x^{2019} + x^3 & \text{to} & x^{2019} \\ x^{2019} + x^3 & \text{to} & x^{2019} + x^3 + x^2 \\ x^{2019} + x^2 + x + 1 & \text{to} & x^{2019} - x^2 - x - 1 \end{array}$$

Let $\mathcal{B}$ be the set of polynomials in $\mathcal{P}$ where all non-leading terms have the same coefficient. There are p polynomials that can be reached from each element of $\mathcal{B}$ in exactly s *swaps*, and there exist 0 polynomials that can be reached from each element of $\mathcal{B}$ in less than s *swaps*.

Compute $p \cdot s$, expressing your answer as a prime factorization.

Solution. $\boxed{2^{2018}3^2}$
Note that there is a bijection between $\mathcal{P}$ and the set of positions reachable in the Tower of Hanoi puzzle, each polynomial representing one position. The state space of the Tower of Hanoi can be further reduced to a graph in the shape of a serpenski triangle and the problem reduces to finding the number of points that have the smallest maximum distance from a vertex of the triangle and the smallest maximum distance, which are 6 and $3 \cdot 2^{2017}$ respectively. We multiply those numbers to obtain $2^{2018}3^2$.

Tiebreaker Round

Problem 1. Let ABC be a triangle $\angle BAC = 60^\circ, \angle ABC = 70^\circ, \angle ACB = 50^\circ$. Let D, E, F be the feet of the altitudes from A, B, C respectively. Suppose $AD = \frac{1}{2}$. Let O be the circumcenter of ABC. Suppose line AO intersects segment BC at point O'. Find AO'.

Solution. $\boxed{1}$
Let $T = EF \cap BC$. Remark $\angle AO'C = 180 - (90 - C) - B = 90 + C - B = B$ as $2B = 90 + C$. Thus $AO' = AB = \frac{AD}{\cos 60} = 1$.

Problem 2. Let r_1, r_2, r_3, r_4 denote the values of the roots of the quartic $x^4 + 2x^3 + 4x^2 - 3x + 5$. Find $\prod_{i=1}^{4}(r_i^3 - 8)$.

Solution. $\boxed{4277}$
Denote the quartic by $p(x)$. Note

$$\prod_{i=1}^{4}(r_i^3 - 8) = \prod_{i=1}^{4}(r_i - 2)\prod_{i=1}^{4}(r_i^2 + 2r_i + 4) = (-1)^4 p(2)\frac{\prod_{i=1}^{4}(3r_i - 5)}{\prod_{i=1}^{4} r_i^2} = p(2) \cdot \frac{(-1)^4 3^4 p(5/3)}{25}$$

Computing we have $p(2) = 47, p(5/3) = \frac{2275}{81}$ so our answer is $47 \cdot 81 \cdot \frac{2275}{81} \cdot \frac{1}{25} = 4277$.

Problem 3. Wendy the tadpole is swimming around the Dawn Pond. She starts at the point $(0, 0)$ and wants to swim to the point $(2\sqrt{3}, 0)$. However, Wendy can only make four jumps of length 1. Let P be the point Wendy reaches after the second jump. Let R be the locus of all such points P over all possible paths Wendy can take. Find the area of region R.

Solution. $\boxed{\frac{4\pi}{3} - 2\sqrt{3}}$
After 2 jumps Wendy must be within distance 2 of $(0, 0)$. Also her ending point is reachable in 2 jumps so P is also within distance 2 of $(2\sqrt{3}, 0)$ Therefore R is the intersection of circles of radius 2, at a distance $2\sqrt{3}$ apart. This area is 2 circular arcs of radius 2 and angle $\frac{\pi}{3}$ minus the the rhombus with sides 2 and long diagonal $2\sqrt{3}$. The rhombus will have the other diagonal of 2. Thus we have $2\pi(2)^2/6 - \frac{1}{2}(2)(2\sqrt{3}) = \frac{4\pi}{3} - 2\sqrt{3}$.

Problem 4. How many integers $0 \le x \le 2019$ satisfy $2019|(x^5 + x^3 + x)$?

Solution. $\boxed{16}$
$x = 0, 2019$ are both solutions, now suppose $x \not\equiv 0 \pmod{2019}$, so $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1) \equiv 0 \pmod{2019}$. $2019 = 3 \cdot 673$ and 673 is prime.

First, suppose 673 does not divide x. If $x \equiv 1, 2 \pmod 3$, then $(x^2 + x + 1)(x^2 - x + 1) \equiv 0 \pmod 3$. If $x \equiv 0 \pmod 3$, then $3|x$. So x can be any residue $\pmod 3$.

Now either we have $x^2 + x + 1 \equiv 0 \pmod{673}$ or $x^2 - x + 1 \equiv 0 \pmod{673}$; if both quantities are 0 (mod 673) then their difference $2x \equiv 0 \pmod{673} \implies x \equiv 0 \pmod{673}$ which leads to none of those quantities being $0 \pmod{673}$.

If $x^2 \pm x + 1 \equiv 0 \pmod{2019}$, then $(2x \pm 1)^2 \equiv -3 \pmod{673}$. We can compute by Quadratic Reciprocity

$$\left(\frac{-3}{673}\right) = \left(\frac{-1}{673}\right)\left(\frac{3}{673}\right) = 1 \cdot \frac{(-1)^{\frac{3-1}{2}\frac{673-1}{2}}}{\left(\frac{673}{3}\right)} = 1$$

We can check that some residue a modulo 2019 cannot be a solution to both $x^2 + x + 1 \equiv 0 \pmod{673}$ and $x^2 - x + 1 \equiv 0 \pmod{673}$ simultaneously, hence there are 4 distinct residues $\pmod{673}$ so that $(x^2 + x + 1)(x^2 - x + 1) \equiv 0 \pmod{673}$.

Next, suppose 673 does divide x. Then $x \equiv 1, 2 \pmod{3}$ as $(x^2 + x + 1)(x^2 - x + 1) \equiv 0 \pmod{3}$. This yields 2 residues $\pmod{3}$.

This leads to $4 \cdot 3 + 1 \cdot 2 = 12 + 2$ nonzero solutions $\pmod{2019}$ by the Chinese Remainder Theorem, so the total answer is $14 + 2 = 14$.

Problem 5. Let $A(0,0)$ and $B(1,0)$ be points in the plane. Let R be the region in the plane such that for any point C in R, $m\angle ACB > 30\circ$. Compute the area of R.

Solution. $\boxed{\frac{5}{3}\pi + \frac{\sqrt{3}}{2}}$

R is the union of two circles of radius 1 centered at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. These circles are the locus of points for which $m\angle ACB = 30°$. A simple construction shows that for any C inside the interior of one of these circles, $m\angle ACB > 30°$, and the contrary for any C outside these circles.

We can split R along the x-axis into two identical regions consisting of a $300\circ$ arc of a circle of radius 1 and an equilateral triangle of side length 1. Thus, the area is

$$2\left(\frac{5}{6}\pi(1^2) + \frac{\sqrt{3}}{4}(1^2)\right) = \frac{5}{3}\pi + \frac{\sqrt{3}}{2}$$