Team Round

Problem 1. Let ABC be an equilateral triangle of side length 6. Points D, E and F are on sides AB, BC, and AC respectively such that $AD = BE = CF = 2$. Let circle O be the circumcircle of DEF, that is, the circle that passes through points D, E, and F. What is the area of the region inside triangle ABC but outside circle O?

Problem 2. Alex, Bob, Charlie, Daniel, and Ethan are five classmates. Some pairs of them are friends. How many possible ways are there for them to be friends such that everyone has at least one friend, and such that there is exactly one loop of friends among the five classmates?

Note: friendship is two-way, so if person x is friends with person y then person y is friends with person x.

Problem 3. A frog is jumping between lattice points on the coordinate plane in the following way: On each jump, the frog randomly goes to one of the 8 closest lattice points to it, such that the frog never goes in the same direction on consecutive jumps. If the frog starts at $(20, 19)$ and jumps to $(20, 20)$, then what is the expected value of the frog's position after it jumps for an infinitely long time?

Problem 4. Let ΔABC be a triangle such that the area $[ABC] = 10$ and $\tan(\angle ABC) = 5$. If the smallest possible value of $\left(\overline{AC}\right)^2$ can be expressed as $-a + b\sqrt{c}$ for positive integers a, b, c, what is $a + b + c$?

Problem 5. A tournament has 5 players and is in round-robin format (each player plays each other exactly once). Each game has a $\frac{1}{3}$ chance of player A winning, a $\frac{1}{3}$ chance of player B winning, and a $\frac{1}{3}$ chance of ending in a draw. The probability that at least one player draws all of their games can be written in simplest form as $\frac{m}{3^n}$ where m, n are positive integers. Find $m + n$.

Problem 6. Compute $\displaystyle\prod_{i=1}^{2019}(2^{2^i} - 2^{2^{i-1}} + 1)$.

Problem 7. Let S be the set of all positive integers n satisfying the following two conditions:

- n is relatively prime to all positive integers less than or equal to $\frac{n}{6}$.
- $2^n \equiv 4 \mod n$

What is the sum of all numbers in S?

Problem 8. Consider an infinite sequence of reals $x_1, x_2, x_3, \ldots$ such that $x_1 = 1, x_2 = \frac{2\sqrt{3}}{3}$ and with the recursive relationship

$$n^2(x_n - x_{n-1} - x_{n-2}) - n(3x_n + 2x_{n-1} + x_{n-2}) + (x_n x_{n-1} x_{n-2} + 2x_n) = 0$$

Find x_{2019}. [1]

Problem 9. Consider a rectangle with length 6 and height 4. A rectangle with length 3 and height 1 is placed inside the larger rectangle such that it is distance 1 from the bottom and leftmost sides of the larger rectangle.

We randomly select one point from each side of the larger rectangle, and connect these 4 points to form a quadrilateral. What is the probability that the smaller rectangle is strictly contained within that quadrilateral?

Problem 10. n players are playing a game. Each player has n tokens. Every turn, two players with at least one token are randomly selected. The player with less tokens gives one token to the player with more tokens. If both players have the same number of tokens, a coin flip decides which player receives a token and which player gives a token. The game ends when one player has all the tokens. If $n = 2019$, suppose the maximum number of turns the game could take to end can be written as $\frac{1}{d}(a \cdot 2019^3 + b \cdot 2019^2 + c \cdot 2019)$ for integers a, b, c, d. Find $\frac{abc}{d}$.

[1] This problem was thrown out due to an error in the problem statement. The correct recursive relationship should have been: $n^2(x_n - x_{n-1} - x_{n-2}) - n(3x_n - 2x_{n-1} - x_{n-2}) + (2x_n - x_n x_{n-1} x_{n-2}) = 0$.