CHMMC 2020-2021

Proof Round

Maximum Possible Score: 40 pts

For multi-part problems, you may solve the parts in any order and cite the assumed result of one part to solve another as long as you do not use circular reasoning.

1. *(5 pts)* Let n be a positive integer, $K = \{1, 2, \ldots, n\}$, and $\sigma : K \to K$ be a function with the property that $\sigma(i) = \sigma(j)$ if and only if $i = j$ (in other words, σ is a *bijection*). Show that there is a positive integer m such that

$$\underbrace{\sigma(\sigma(\ldots\sigma(i)\ldots))}_{m \text{ times}} = i$$

for all $i \in K$.

2. *(5 pts)* For some positive integer n, let $P(x)$ be an nth degree polynomial with real coefficients. *Note: you may cite, without proof, the Fundamental Theorem of Algebra, which states that every non-constant polynomial with complex coefficients has a complex root.*

(a) *(2 pts)* Show that there is an integer $k \geq \frac{n}{2}$ and a sequence of non-constant polynomials with real coefficients $Q_1(x), Q_2(x), \ldots, Q_k(x)$ such that

$$P(x) = \prod_{i=1}^{k} Q_i(x).$$

(b) *(1 pt)* If n is odd, then show that $P(x)$ has a real root.

(c) *(2 pts)* Let a and b be real numbers, and let m be a positive integer. If $\zeta = a + bi$ is a nonreal root of $P(x)$ of multiplicity m, then show that $\overline{\zeta} = a - bi$ is a nonreal root of $P(x)$ of multiplicity m.

3. *(6 pts)* Find all positive integers $n \geq 3$ such that there exists a permutation $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ such that $a_1, 2a_2, \ldots, na_n$ can be rearranged into an arithmetic progression.

4. *(7 pts)* Fix a positive integer n. Pick $4n$ equally spaced points on a circle and color them alternately blue and red. You use n blue chords to pair the $2n$ blue points, and you use n red chords to pair the $2n$ red points. If some blue chord intersects some other red chord, then such a pair of chords is called a "good pair."

(a) *(1 pts)* For the case $n = 3$, explicitly show that there are at least 3 distinct ways to pair the $2n$ blue points and the $2n$ red points such that there are a total of 3 good pairs (2 configurations of chord pairings are *not* considered distinct if one of them can be "rotated" to the other).

(b) *(6 pts)* Now suppose that n is arbitrary. Find, with proof, the minimum number of good pairs under all possible configurations of chord pairings.

5. *(8 pts)* Let n be a positive integer, and let a, b, c be real numbers.

(a) *(2 pts)* Given that $a\cos x + b\cos 2x + c\cos 3x \geq -1$ for all reals x, find, with proof, the maximum possible value of $a + b + c$.

(b) *(6 pts)* Let f be a degree n polynomial with real coefficients. Suppose that $|f(z)| \leq \left|f(z) + \frac{2}{z}\right|$ for all complex z lying on the unit circle. Find, with proof, the maximum possible value of $f(1)$.

6. *(9 pts)* Let ABC be a triangle with circumcenter O. The interior bisector of $\angle BAC$ intersects BC at D. Circle ω_A is tangent to segments AB and AC and internally tangent to the circumcircle of ABC at the point P. Let E and F be the respective points at which the B-excircle and C-excircle of ABC are tangent to AC and AB. Suppose that lines BE and CF pass through a common point N on the circumcircle of AEF.

Note: for a triangle ABC, the A-excircle is the circle lying outside triangle ABC that is tangent to side BC and the extensions of sides AB, AC. The B, C-excircles are defined similarly.

(a) *(7 pts)* Prove that the circumcircle of PDO passes through N.

(b) *(2 pts)* Suppose that $\frac{PD}{BC} = \frac{2}{7}$. Find, with proof, the value of $\cos(\angle BAC)$.