# CHMMC 2020-2021

## Team Round Solutions

**1.** A unit circle is centered at $(0, 0)$ on the $(x, y)$ plane. A regular hexagon passing through $(1, 0)$ is inscribed in the circle. Two points are randomly selected from the interior of the circle and horizontal lines are drawn through them, dividing the hexagon into at most three pieces. The probability that each piece contains exactly two of the hexagon's original vertices can be written as

$$\frac{2\left(\frac{m\pi}{n}+\frac{\sqrt{p}}{q}\right)^2}{\pi^2}$$

for positive integers $m$, $n$, $p$, and $q$ such that $m$ and $n$ are relatively prime and $p$ is squarefree. Find $m + n + p + q$.

**Solution:** 11

The line passing through the first point must be in between the "top" and "bottom" segments of the hexagon. The total area in which we can choose this point from equals the area of the hexagon plus the area of 4 $60^\circ$ circular segments. This is 2 unit equilateral triangles plus 4 $60^\circ$ circular sectors, an area of $2 \cdot (\frac{\sqrt{3}}{4} + \frac{\pi}{3})$. Hence, the probability that the line passing through the first point satisfies the aforementioned condition is just $\frac{2\cdot(\frac{\sqrt{3}}{4}+\frac{\pi}{3})}{\pi}$.

WLOG, assume that the line passing through the first point is below the horizontal long diagonal of the hexagon. Then, the line passing through the second point must be in between the "top" segment of the hexagon and the horizontal long diagonal. We see that the possible area to choose the second point is half the area of the hexagon plus 2 $60^\circ$ circular segments. This is a unit equilateral triangle plus 2 $60^\circ$ circular sectors, an area of $(\frac{\sqrt{3}}{4} + \frac{\pi}{3})$. Hence, the probability that the line passing through the second point satisfies the aforementioned condition is just $\frac{\frac{\sqrt{3}}{4}+\frac{\pi}{3}}{\pi}$.

The overall probability of success is the product of the two probabilities, or $\frac{2\cdot(\frac{\sqrt{3}}{4}+\frac{\pi}{3})^2}{\pi^2}$. Hence, the answer is $3 + 4 + 1 + 3 = \boxed{11}$.

**2.** Find the smallest positive integer $k$ such that there is exactly one prime number of the form $kx+60$ for the integers $0 \le x \le 10$.

**Solution:** 17

First note that if there is a common factor $d = \gcd(k, 60)$, then $d$ will divide $kx + 60$. Thus, we want $\gcd(k, 60) = 1$. Since 60 is divisible by 2, 3, 4, and 5 we start with $k = 1$ and then primes.

Similarly we also want $x$ and 60 to be relatively prime. Therefore it is only necessary to check $x = 1, 7$. One can compute

$$k = 1 \longrightarrow kx + 60 = 61, 67.$$
$$k = 7 \longrightarrow kx + 60 = 67, 109.$$
$$k = 11 \longrightarrow kx + 60 = 71, 137.$$
$$k = 13 \longrightarrow kx + 60 = 73, 151.$$
$$k = 17 \longrightarrow kx + 60 = 77, 179.$$

For the first 4, we find two prime numbers each. However, for $k = 17$ we note that 77 is composite while 179 is prime. Hence, $k = \boxed{17}$ is the answer.

**3.** For any nonnegative integer $n$, let $S(n)$ be the sum of the digits of $n$. Let $K$ be the number of nonnegative integers $n \leq 10^{10}$ that satisfy the equation

$$S(n) = (S(S(n)))^2.$$

Find the remainder when $K$ is divided by 1000.

**Solution:** 632

Since $n \leq 10^{10}$, we have that $S(n) \leq 90$. Testing possible perfect square values of $S(n)$, we see that the only possible values of $S(n)$ that satisfy $S(n) = (S(S(n)))^2$ are $S(n) = 0, 1, 81$. For $S(n) = 0$, only $n = 0$ works. For $S(n) = 1$, the values $n = 10^0, 10^1, \ldots, 10^{10}$ work, a total of 11 cases. For $S(n) = 81$, we want 10 digits, each of which is at most, to sum to 81. This is equivalent to a ball-and-urn problem with 9 balls and 10 urns, where each urn represents a digits-place and $k$ balls in an urn represents a digit value of $9 - k$. The number of ways to put the balls into urns equals $\binom{9+10-1}{10-1}$. Hence, the number of integers $n$ is $12 + \binom{18}{9}$; when divided by 1000, this yields $\boxed{632}$.

**4.** Select a random real number $m$ from the interval $(\frac{1}{6}, 1)$. A track is in the shape of an equilateral triangle of side length 50 feet. Ch, Hm, and Mc are all initially standing at one of the vertices of the track. At the time $t = 0$, the three people simultaneously begin walking around the track in clockwise direction. Ch, Hm, and Mc walk at constant rates of $2, 3$, and 4 feet per second, respectively. Let $T$ be the set of all positive real numbers $t_0$ satisfying the following criterion:

*If we choose a random number $t_1$ from the interval $[0, t_0]$, the probability that the three people are on the same side of the track at the time $t = t_1$ is precisely $m$.*

The probability that $|T| = 17$ (i.e., $T$ has precisely 17 elements) equals $\frac{p}{q}$, where $p$ and $q$ are relatively prime positive integers. Find $p + q$.

**Solution:** 10585

Denote Ch, Hm, and Mc by C, H, and M respectively. Label the sides of the triangle by $1, 2, 3$ in clockwise order.

Note that C is at side 1 at times $[0, 25] \cup [75, 100] \cup [150, 175] \cup \cdots$, H is at side 1 at times $[0, \frac{50}{3}] \cup [50, 50 + \frac{50}{3}] \cup [100, 100 + \frac{50}{3}] \cup [150, 150 + \frac{50}{3}] \cup \cdots$, and M is at side 1 at times $[0, \frac{50}{4}] \cup [\frac{150}{4}, \frac{150}{4} + \frac{50}{4}] \cup [\frac{300}{4}, \frac{300}{4} + \frac{50}{4}] \cup [\frac{450}{4}, \frac{450}{4} + \frac{50}{4}] \cup [\frac{600}{4}, \frac{600}{4} + \frac{50}{4}] \cup \cdots$. So, C, H, and M are all at side 1 at times $[150k, 150k + \frac{25}{2}]$ for integers $k \geq 0$, noting the pattern is periodic $\pmod{150}$ seconds.

Note that C is at side 2 at times $[25, 50] \cup [100, 125] \cup [175, 200] \cup \cdots$, H is at side 2 at times $[\frac{50}{3}, 2 \cdot \frac{50}{3}] \cup [50 + \frac{50}{3}, 50 + 2 \cdot \frac{50}{3}] \cup [100 + \frac{50}{3}, 100 + 2 \cdot \frac{50}{3}] \cup [150 + \frac{50}{3}, 150 + 2 \cdot \frac{50}{3}] \cup \cdots$, and M is at side 2 at times $[\frac{50}{4}, 2 \cdot \frac{50}{4}] \cup [\frac{150}{4} + \frac{50}{4}, \frac{150}{4} + 2 \cdot \frac{50}{4}] \cup [\frac{300}{4} + \frac{50}{4}, \frac{300}{4} + 2 \cdot \frac{50}{4}] \cup [\frac{450}{4} + \frac{50}{4}, \frac{450}{4} + 2 \cdot \frac{50}{4}] \cup [\frac{600}{4} + \frac{50}{4}, \frac{600}{4} + 2 \cdot \frac{50}{4}] \cup \cdots$. We can check that C, H, M are never all at side 2, noting the pattern is periodic $\pmod{150}$ seconds.

Note that C is at side 3 at times $[50, 75] \cup [125, 150] \cup [200, 225] \cup \cdots$, H is at side 3 at times $[2 \cdot \frac{50}{3}, 3 \cdot \frac{50}{3}] \cup [50 + 2 \cdot \frac{50}{3}, 50 + 3 \cdot \frac{50}{3}] \cup [100 + 2 \cdot \frac{50}{3}, 100 + 3 \cdot \frac{50}{3}] \cup [150 + 2 \cdot \frac{50}{3}, 150 + 3 \cdot \frac{50}{3}] \cup \cdots$, and M is at side 3 at times $[2 \cdot \frac{50}{4}, 3 \cdot \frac{50}{4}] \cup [\frac{150}{4} + 2 \cdot \frac{50}{4}, \frac{150}{4} + 3 \cdot \frac{50}{4}] \cup [\frac{300}{4} + 2 \cdot \frac{50}{4}, \frac{300}{4} + 3 \cdot \frac{50}{4}] \cup [\frac{450}{4} + 2 \cdot \frac{50}{4}, \frac{450}{4} + 3 \cdot \frac{50}{4}] \cup [\frac{600}{4} + 2 \cdot \frac{50}{4}, \frac{600}{4} + 3 \cdot \frac{50}{4}] \cup \cdots$. So, C, H, and M are all at side 3 at times $[150k - \frac{25}{2}, 150k]$ for integers $k \geq 1$, noting the pattern is periodic $\pmod{150}$ seconds.

If $m \in (\frac{2(k+1)+1}{12(k+1)+1}, \frac{2k+1}{12k+1}]$, then by a continuity argument, there is exactly one element of $T$ in the intervals $(150(i-1) + \frac{25}{2}, 150i - \frac{25}{2})$ for all $i = 1, \ldots, k+1$ and an element of $T$ in the intervals $[\frac{1}{6}, \frac{2i+1}{12i+1}]$ for all $i = 1, \ldots, k$, and there are no other elements of $T$.

Hence, the probability that $|T| = 17$ equals the probability that $m \in (\frac{2 \cdot 9+1}{12 \cdot 9+1}, \frac{2 \cdot 8+1}{12 \cdot 8+1}]$, so the requested probability is

$$\left(\frac{17}{97} - \frac{19}{109}\right) \Big/ \left(\frac{5}{6}\right) = \frac{6}{5} \cdot \frac{17 \cdot 109 - 19 \cdot 107}{97 \cdot 109} = \frac{12}{97 \cdot 109}.$$

This computation is made easier by noting that $(12(k+1)+1)(2k+1) - (12k+1)(2(k+1)+1) = 12 - 2 = 10$ in general. Thus, the answer is $\boxed{10585}$.

**5.** Thanos establishes 5 settlements on a remote planet, randomly choosing one of them to stay in, and then he randomly builds a system of roads between these settlements such that each settlement has exactly one outgoing (unidirectional) road to another settlement. Afterwards, the Avengers randomly choose one of the 5 settlements to teleport to. Then, they (the Avengers) must use the system of roads to travel from one settlement to another. The probability that the Avengers can find Thanos can be written as $\frac{m}{n}$ for relatively prime positive integers $m$ and $n$. Find $m+n$.

**Solution:** 263

The Avengers have a $\frac{1}{5}$ chance of immediately finding Thanos. Otherwise, we can model the system of roads as a random walk of maximum length 4 on the 5 vertices of a complete graph. In each "step" of the random walk, we end the walk in success if the Avengers reach the vertex at which Thanos resides (a $\frac{1}{4}$ probability each time), and we end the walk in failure if the Avengers revisit a vertex, as this implies the Avengers are stuck in a "loop" of directed roads where Thanos does not exist. Otherwise, we continue the random walk.

Observe that step $n$ has a $\frac{1}{4}$ chance of success, an $\frac{n-1}{4}$ chance of failure, and a $\frac{4-n}{4}$ chance of continuation. Therefore, the chance of succeeding on step $n$ can be given by

$$\frac{1}{4} \cdot \prod_{k=1}^{n-1} \frac{4-k}{4}.$$

Thus, the overall probability that the Avengers can find Thanos is

$$\frac{1}{5} + \frac{4}{5} \cdot \frac{1}{4} \left( \sum_{j=1}^{4} \prod_{k=1}^{j-1} \frac{4-k}{4} \right) = \frac{103}{160}.$$

Hence, the answer is $\boxed{263}$.

**6.** Suppose that

$$\prod_{n=1}^{\infty} \left( \frac{1 + i\cot\left(\frac{n\pi}{2n+1}\right)}{1 - i\cot\left(\frac{n\pi}{2n+1}\right)} \right)^{\frac{1}{n}} = \left(\frac{p}{q}\right)^{i\pi},$$

where $p$ and $q$ are relatively prime positive integers. Find $p+q$.
*Note: for a complex number $z = re^{i\theta}$ for reals $r > 0, 0 \le \theta < 2\pi$, we define $z^n = r^n e^{i\theta n}$ for all positive reals $n$.*

**Solution:** 5

Let $\alpha = \frac{p}{q}$. Since $1 + i\cot\left(\frac{n\pi}{2n+1}\right)$ and $1 - i\cot\left(\frac{n\pi}{2n+1}\right)$ clearly have equal magnitude, their quotient is in the form $e^{i\theta}$, where $\theta$ is the difference between their arguments. If we examine the right triangle with leg lengths 1 and $\cot\left(\frac{n\pi}{2n+1}\right)$, we can indeed see that the measure of the non-right angle adjacent to the leg of length 1 is $\frac{\pi}{2(2n+1)}$, so $\arg\left(1+i\cot\left(\frac{n\pi}{2n+1}\right)\right) = \frac{\pi}{2(2n+1)}$ and $\arg\left(1-i\cot\left(\frac{n\pi}{2n+1}\right)\right) = -\frac{\pi}{2(2n+1)}$. Thus, the desired result is

$$\prod_{n=1}^{\infty}\left(e^{\frac{i\pi}{2n+1}}\right)^{\frac{1}{n}} = \prod_{n=1}^{\infty} e^{\frac{i\pi}{n(2n+1)}} = e^{Si\pi},$$

where

$$S = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \cdots = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right).$$

However, the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges to $\ln 2$, so in fact

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = 1 - \ln 2 \implies S = 2 - 2\ln 2.$$

Therefore,

$$e^{Si\pi} = e^{2i\pi - 2\ln(2)i\pi} = e^{-2\ln(2)i\pi} = \left(\frac{1}{4}\right)^{i\pi} \implies \alpha = \frac{1}{4}.$$

Thus, the answer is $\boxed{5}$.

Remark: we can obtain other $\alpha$ that satisfy the specified equation by multiplying/dividing $\frac{1}{4}$ by $e^2$, but those values of $\alpha$ are not rational.

**7.** For any positive integer $n$, let $f(n)$ denote the sum of the positive integers $k \leq n$ such that $k$ and $n$ are relatively prime. Let $S$ be the sum of $\frac{1}{f(m)}$ over all positive integers $m$ that are divisible by at least one of 2, 3, or 5, and whose prime factors are only 2, 3, or 5. Then $S = \frac{p}{q}$ for relatively prime positive integers $p$ and $q$. Find $p+q$.

**Solution:** 7291

Remark: for any positive integer $n > 1$, where $\phi$ denotes the Euler Totient function,

$$f(n) = \sum_{1\leq x\leq n, \gcd(x,n)=1} x = \phi(n)\cdot\frac{n}{2}$$

since $\gcd(x,n) = 1 \iff \gcd(n-x,n) = 1$. Let $\mathcal{I}$ denote the set of positive integers $m$ that are divisible by at least one of 2, 3, or 5, and whose prime factors are only 2, 3, or 5. Thus,

$$S = \sum_{m\in\mathcal{I}} \frac{1}{f(m)} = 2\sum_{m\in\mathcal{I}} \frac{1}{m\phi(m)}$$

Then, by the multiplicativity of $\phi$ on prime powers, we see that $S = 2(-1 + Q(2)Q(3)Q(5))$ where for a prime $p$,

$$Q(p) = \frac{1}{\phi(1)\cdot 1} + \frac{1}{\phi(p)\cdot p} + \frac{1}{\phi(p^2)\cdot p^2} + \cdots$$

For any $k \geq 1$ we have $p^k\phi(p^k) = p^{k-1}(p-1)p^k = p^{2k-1}(p-1)$, so

$$Q(p) = 1 + \frac{1}{p-1}\cdot\frac{1/p}{1-1/p^2} = 1 + \frac{p}{(p-1)(p^2-1)}$$

Therefore,

$$S = 2\left(\frac{5}{3}\cdot\frac{19}{16}\cdot\frac{101}{96} - 1\right) = \frac{9595}{48^2} - 2 = \frac{4987}{2304}.$$

The answer is $\boxed{7291}$.

**8.** 15 ladies and 30 gentlemen attend a luxurious party. At the start of the party, each one of the ladies shakes hands with a random gentleman. At the end of the party, each of the ladies shakes hands with another random gentleman. A lady may shake hands with the same gentleman twice (first at the start and then at the end of the party), and no two ladies shake hands with the same gentleman at the same time.

Let $m$ and $n$ be relatively prime positive integers such that $\frac{m}{n}$ is the probability that the collection of ladies and gentlemen that shook hands at least once can be arranged in a single circle such that each lady is directly adjacent to someone if and only if she shook hands with that person. Find the remainder when $m$ is divided by 10000.

**Solution:** 1401

**Case 1:** 15 gentlemen shake hands twice.

If we line up every lady with the gentleman she shook hands with at the start of the party, then the number of ways the ladies could have shaken hands with gentlemen at the end of the party describes the set of permutations of 15 gentlemen. In a given permutation of 15 gentlemen, every $n$-cycle describes a circle formed by alternating $n$ ladies and $n$ gentlemen. Thus, the 15 ladies and 15 gentlemen can form a circle if and only if the described permutation of 15 gentlemen is a 15-cycle. There are 14! 15 cycles and a total of 15! permutations, so the probability that the circle can be formed in this case is $\frac{1}{15}$. The probability that this case occurs is $\frac{1}{\binom{30}{15}}$.

**Case 2:** $1 \leq k \leq 15$, $15 + k$ gentlemen shake hands.

Once again, we line up every lady with the gentleman she shook hands with at the start of the party. We may assume WLOG that the last $k$ ladies shake hands at the end of the party with the $k$ gentlemen that did note shake hands at the start of the party. Consider the graph of 15 ladies and $15 + k$ gentlemen such that two nodes are connected if and only if the corresponding lady and gentleman shook hands. We call 2 ladies "linked" if and only if we can trace a path of connected nodes between them. Observe that the 15 ladies and $15 + k$ gentlemen can form the desired circle if and only if every lady is linked to one of the last $k$ ladies. Then, the number of ways for this to happen is a bijection to the stars and bars problem with $15 - k$ distinct stars and $k - 1$ identical bars, a total of $(15 - k)!\binom{14}{k-1}$ ways. The total number of ways the first $15 - k$ ladies could have chosen the 15 available men (the ones which shook hands at the start of the party) is $\frac{15!}{k!}$. Thus, we see that the probability that the circle can be formed in this case is $\frac{(15-k)!\binom{14}{k-1}k!}{15!} = \frac{k}{15}$. The probability that this case occurs, for each $k$, is $\frac{\binom{15}{k}\binom{15}{15-k}}{\binom{30}{15}}$.

Letting $P$ be the desired probability, we see that

$$P = \frac{\sum_{n=0}^{15} \frac{n}{15}\binom{15}{n}\binom{15}{15-n} + \frac{1}{15}}{\binom{30}{15}} = \frac{\frac{1}{2}\binom{30}{15} + \frac{1}{15}}{\binom{30}{15}} = \frac{\frac{15}{2}\binom{30}{15} + 1}{15\binom{30}{15}}$$

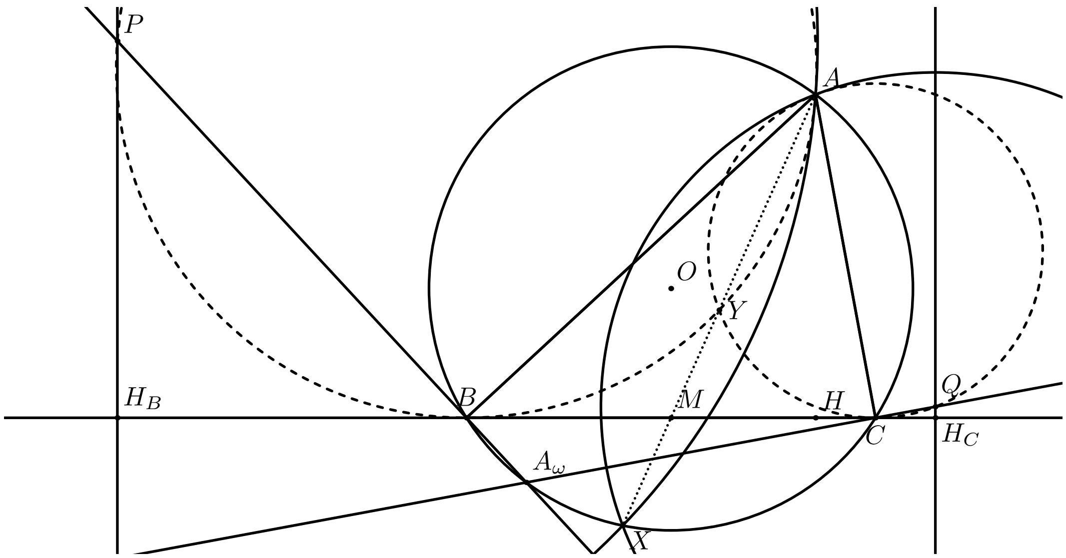
by symmetry of $\frac{n}{15}$ terms and Vandermonde's identity. We observe that $15\binom{30}{15} = 2^4 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29$ and $\frac{15}{2}\binom{30}{15} + 1 = 2^3 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 + 1$ so the numerator and denominator are clearly coprime. Observe that $2 \cdot 3^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \equiv 14 \pmod{100}$, so the remainder when $2^3 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 + 1$ is divided by 10000 is $\boxed{1401}$.

**9.** Triangle $ABC$ has circumcenter $O$ and circumcircle $\omega$. Let $A_\omega$ be the point diametrically opposite $A$ on $\omega$, and let $H$ be the foot of the altitude from $A$ onto $BC$. Let $H_B$ and $H_C$ be the reflections of $H$ over $B$ and $C$, respectively. Point $P$ is the intersection of line $A_\omega B$ and the perpendicular of

$BC$ at point $H_B$, and point $Q$ is the intersection of line $A_\omega C$ and the perpendicular of $CB$ at point $H_C$. The circles $\omega_1$ and $\omega_2$ have the respective centers $P$ and $Q$ and respective radii $PA$ and $QA$. Suppose that $\omega$, $\omega_1$, and $\omega_2$ intersect at another common point $X$. If $AO = \frac{\sqrt{105}}{5}$ and $AX = 4$, then $|AB - CA|^2$ can be written as $m - n\sqrt{p}$ for positive integers $m$ and $n$ and squarefree positive integer $p$. Find $m + n + p$.

*Note: the reflection of a point $P$ over another point $Q \neq P$ is the point $P'$ such that $Q$ is the midpoint of $P$ and $P'$.*

**Solution:** 44




Since $AA_\omega$ is a diameter of $\omega$, we see that $AB \perp PA_\omega$ and $AC \perp QA_\omega$. Hence, $B$ is the midpoint of the chord in $\omega_1$ defined by line $AB$, and $C$ is the midpoint of the chord in $\omega_2$ defined by line $AC$. Consider the homothety $\mathcal{H}(A, \frac{1}{2})$ which sends $P$ to $P'$, $Q$ to $Q'$, $\omega_1$ to $\gamma_1$, and $\omega_2$ to $\gamma_2$. Observe that $\gamma_1$ passes through point $B$, $\gamma_2$ passes through point $C$, and $\gamma_1$ and $\gamma_2$ intersect at point $Y$ the midpoint of $AX$. Furthermore, since $P$ lies on the reflection of line $AH$ over point $B$, and $Q$ lies on the reflection of line $AH$ over point $C$, observe that $P'B \perp BC$ and $Q'C \perp CB$. Thus, $\gamma_1$ is tangent to $BC$ at point $B$, and $\gamma_2$ is tangent to $CB$ at point $C$. Thus, $AY$ bisects $BC$ at point $M$. By power of a point,

$$MX \cdot MA = MB \cdot MC = MB^2 = MY \cdot MA \implies MX = MY = \frac{YA}{2} = 1.$$

We see that $MB = MC = \sqrt{3}$ and $Y$ is the centroid of $\triangle ABC$. Furthermore, $OY \perp AX$, so applying the Pythagorean Theorem on right $\triangle OYX$ gives us $OY = \sqrt{\frac{105}{5^2} - 2^2} = \frac{\sqrt{5}}{5}$. Apply the Pythagorean Theorem again on $\triangle OYM$ to obtain $OM = \sqrt{\frac{5}{5^2} + 1} = \frac{\sqrt{30}}{5}$. Note that $OM$ is the perpendicular bisector of $BC$. Thus, the distance from $Y$ to $OM$ is $\frac{OY \cdot YM}{OM} = \sqrt{\frac{1}{6}}$, and the distance from $Y$ to $BC$ is $\frac{YM^2}{OM} = \sqrt{\frac{5}{6}}$. By the Pythagorean Theorem,

$$BY = \sqrt{\left(\sqrt{3} \pm \sqrt{\frac{1}{6}}\right)^2 + \frac{5}{6}}, CY = \sqrt{\left(\sqrt{3} \mp \sqrt{\frac{1}{6}}\right)^2 + \frac{5}{6}}.$$

However, $BC$ is tangent to $\gamma_1, \gamma_2$ and $AM$ is a common secant of these two circles. This implies that $\triangle MBY \sim \triangle MAB$ and $\triangle MCY \sim \triangle MAC$, both of similitude ratio $\frac{1}{\sqrt{3}}$. Hence, we have that $|AB - CA|^2 = \left(\sqrt{3}\right)^2 |BY - CY|^2 = 24 - 6\sqrt{14}$. The answer is $\boxed{44}$.

**10.** Let $\omega$ be a nonreal 47th root of unity. Suppose that $\mathcal{S}$ is the set of polynomials of degree at most 46 and coefficients equal to either 0 or 1. Let $N$ be the number of polynomials $Q \in \mathcal{S}$ such that

$$\sum_{j=0}^{46} \frac{Q(\omega^{2j}) - Q(\omega^j)}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = 47.$$

The prime factorization of $N$ is $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$ where $p_1, \dots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_s$ are positive integers. Compute $\sum_{j=1}^{s} p_j \alpha_j$.

**Solution:** 107

For a number $n \in \{0, 1, \dots, 46\}$, we see that

$$\sum_{j=1}^{46} \frac{\omega^{nj}}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = \sum_{j=1}^{46} \frac{\omega^{nj}(\omega^j - 1)}{\omega^{5j} - 1} = \sum_{j=1}^{46} \frac{\omega^{nj}(\omega^{95j} - 1)}{\omega^{5j} - 1} = \sum_{j=1}^{46} \sum_{k=0}^{18} \omega^{(5k+n)j}.$$

Observe that $\sum_{j=1}^{46} \omega^{(5k+n)j} = 46$ if $5k+n$ is a multiple of 47 (i.e., $\omega^{5k+n} = 1$) and $\sum_{j=1}^{46} \omega^{(5k+n)j} = -1$ otherwise (since the sum of all the nonreal roots of unity is $-1$). Since $k$ varies from 0 to 18, we see from inspection that the congruence $5k + n \equiv 0 \pmod{47}$ has exactly one solution in $k$ for $n = 0$ or $n \equiv 2, 4 \pmod{5}$ (call this Condition 1), and $5k + n \equiv 0 \pmod{47}$ has no solutions in $k$ for nonzero $n \equiv 1, 3, 0 \pmod{5}$ (call this Condition 2). Thus,

$$\sum_{j=0}^{46} \frac{\omega^{nj}}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = \frac{1}{5} + \sum_{j=1}^{46} \sum_{k=0}^{18} \omega^{(5k+n)j} = \begin{cases} \frac{141}{5} & \text{if } n \text{ satisfies Condition 1} \\ -\frac{94}{5} & \text{if } n \text{ satisfies Condition 2} \end{cases}.$$

Remark that these formulas hold for when $0 \le n \le 46$.

Denote $f(n) = 2n$ if $0 \le n \le 23$, and $2n - 47$ if $24 \le n \le 46$. Remark $\omega^{2nj} = \omega^{f(n)j}$ for any $0 \le j \le 46$. Then, if we let

$$Z(n) = \sum_{j=0}^{46} \frac{\omega^{f(n)j} - \omega^{nj}}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1},$$

we see that $Z(n) = 47$ if $f(n)$ satisfies Condition 1 and $n$ satisfies Condition 2, $Z(n) = 0$ if $f(n)$ and $n$ satisfy the same of Condition 1 and 2, and $Z(n) = -47$ if $f(n)$ satisfies Condition 2 and $n$ satisfies Condition 1. Testing each value of $n$, we see that

$$Z(n) = \begin{cases} 47 & n \equiv 4 \pmod{5} \\ -47 & n \equiv 1 \pmod{5} \text{ for } n \le 21, n \equiv 3 \pmod{5} \text{ for } n \ge 28 \\ 0 & \text{otherwise} \end{cases}.$$

We have 9 values of $n$ such that $Z(n) = 47$, 29 values of $n$ such that $Z(n) = 0$, and 9 values of $n$ such that $Z(n) = -47$. Observe that polynomials $Q \in \mathcal{S}$ are just additive constructions of $\omega^n$ terms. Indeed, if we let $a_n$ be the coefficient of $x^n$ in the expansion of $Q(x)$, we see that

$$\sum_{j=0}^{46} \frac{Q(\omega^{2j}) - Q(\omega^j)}{\omega^{4j} + \omega^{3j} + \omega^{2j} + \omega^j + 1} = \sum_{0 \le n \le 46 \,:\, a_n = 1} Z(n).$$

Thus, the problem reduces to finding the number of subsets $K \subseteq \{0, 1, \ldots, 46\}$ such that

$$\sum_{n \in K} Z(n) = 47.$$

We can choose to include or not to include any $n$ such that $Z(n) = 0$, giving us $2^{29}$ options here. We must choose exactly one additional $n$ satisfying $Z(n) = 47$ than $n$ satisfying $Z(n) = -47$, giving us $\sum_{j=0}^{8} \binom{9}{j+1}\binom{9}{j} = \binom{18}{10}$ options by Vandermonde's identity. Therefore, the total number of subsets is $\binom{18}{10} \cdot 2^{29} = 2^{30} \cdot 3^2 \cdot 11 \cdot 13 \cdot 17$, giving an answer of $\boxed{107}$.