CHMMC 2021-2022

Rules

1. You have 75 minutes to complete the test.
2. **No collaboration, computers, calculators, or other outside aid is permitted** besides what is listed below.
3. You are, however, permitted to use ruler and compass but **not** a protractor.
4. You can message an organizer **individually** in Slack to ask *only clarifying questions*. You may not receive a response if the question may help you solve the problems.
5. The top 20 individuals will be recognized, and the top 10 will receive trophies. To break ties and determine the ranking of the top 20 individuals we will use a tiebreaker round.
6. All the answers to each problem are integers.

Individual Round

Problem 1. Fleming has a list of 8 mutually distinct integers between 90 to 99, inclusive. Suppose that the list has median 94, and that it contains an even number of odd integers. If Fleming reads the numbers in the list from smallest to largest, then determine the sixth number he reads.

Problem 2. Find the number of ordered pairs (x, y) of three digit base-10 positive integers such that $x - y$ is a positive integer, and there are no borrows in the subtraction $x - y$. *For example, the subtraction on the left has a borrow at the tens digit but not at the units digit, whereas the subtraction on the right has no borrows.*

$$\begin{array}{r} 4\ 7\ 2 \\ -\ 1\ 9\ 1 \\ \hline 2\ 8\ 1 \end{array} \qquad\qquad \begin{array}{r} 3\ 7\ 9 \\ -\ 2\ 6\ 3 \\ \hline 1\ 1\ 6 \end{array}$$

Problem 3. Evaluate

$$1\cdot 2\cdot 3 - 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 - 4\cdot 5\cdot 6 + \cdots + 2017\cdot 2018\cdot 2019 - 2018\cdot 2019\cdot 2020 + 1010\cdot 2019\cdot 2021.$$

Problem 4. Find the number of ordered pairs of integers (a, b) such that

$$\frac{ab + a + b}{a^2 + b^2 + 1}$$

is an integer.

Problem 5. Lin Lin has a 4×4 chessboard in which every square is initially empty. Every minute, she chooses a random square C on the chessboard, and places a pawn in C if it is empty. Then, regardless of whether C was previously empty or not, she then immediately places pawns in all empty squares a king's move away from C. The expected number of minutes before the entire chessboard is occupied with pawns equals $\frac{m}{n}$ for relatively prime positive integers m, n. Find $m + n$.

A king's move, in chess, is one square in any direction on the chessboard: horizontally, vertically, or diagonally.

Problem 6. Let $P(x) = x^5 - 3x^4 + 2x^3 - 6x^2 + 7x + 3$ and $\alpha_1, \ldots, \alpha_5$ be the roots of $P(x)$. Compute

$$\prod_{k=1}^{5} (\alpha_k^3 - 4\alpha_k^2 + \alpha_k + 6).$$

Problem 7. Rectangle $AXCY$ with a longer length of 11 and square $ABCD$ share the same diagonal $\overline{AC}$. Assume B, X lie on the same side of $\overline{AC}$ such that triangle BXC and square $ABCD$ are non-overlapping. The maximum area of BXC across all such configurations equals $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m + n$.

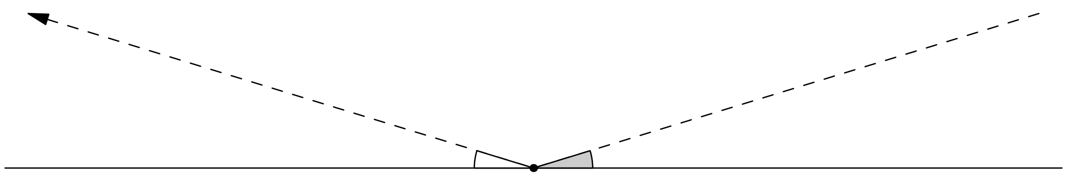
Problem 8. Earl the electron is currently at $(0, 0)$ on the Cartesian plane and trying to reach his house at point $(4, 4)$. Each second, he can do one of three actions: move one unit to the right, move one unit up, or teleport to the point that is the reflection of its current position across the line $y = x$. Earl cannot teleport in two consecutive seconds, and he stops taking actions once he reaches his house.

Earl visits a chronologically ordered sequence of distinct points $(0, 0), \ldots, (4, 4)$ due to his choice of actions. This is called an *Earl-path*. How many possible such Earl-paths are there?

Problem 9. Let $P(x)$ be a degree-2022 polynomial with leading coefficient 1 and roots $\cos(\frac{2\pi k}{2023})$ for $k = 1, \ldots, 2022$ (note $P(x)$ may have repeated roots). If $P(1) = \frac{m}{n}$ where m and n are relatively prime positive integers, then find the remainder when $m + n$ is divided by 100.

Problem 10. A randomly shuffled standard deck of cards has 52 cards, 13 of each of the four suits. There are 4 Aces and 4 Kings, one of each of the four suits. One repeatedly draws cards from the deck until one draws an Ace. Given that the first King appears before the first Ace, the expected number of cards one draws after the first King and before the first Ace is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Problem 11. The following picture shows a beam of light (dashed line) reflecting off a mirror (solid line). The *angle of incidence* is marked by the shaded angle; the *angle of reflection* is marked by the unshaded angle.



The sides of a unit square $ABCD$ are magically distorted mirrors such that whenever a light beam hits any of the mirrors, the measure of the angle of incidence between the light beam and the mirror is a positive real constant θ degrees greater than the measure of the angle of reflection between the light beam and the mirror. A light beam emanating from A strikes $\overline{CD}$ at W_1 such that $2DW_1 = CW_1$, reflects off of $\overline{CD}$ and then strikes $\overline{BC}$ at W_2 such that $2CW_2 = BW_2$, reflects off of $\overline{BC}$, etc. To this end, denote W_i the ith point at which the light beam strikes $ABCD$.

As i grows large, the area of $W_iW_{i+1}W_{i+2}W_{i+3}$ approaches $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Problem 12. For any positive integer m, define $\varphi(m)$ the number of positive integers $k \le m$ such that k and m are relatively prime. Find the smallest positive integer N such that $\sqrt{\varphi(n)} \ge 22$ for any integer $n \ge N$.

Problem 13. Let n be a fixed positive integer, and let $\{a_k\}$ and $\{b_k\}$ be sequences defined recursively by

$$\begin{aligned} &a_1 = b_1 = n^{-1} \\ &a_j = j(n-j+1)a_{j-1},\ j > 1 \\ &b_j = nj^2 b_{j-1} + a_j,\ j > 1 \end{aligned}$$

When $n = 2021$, then $a_{2021} + b_{2021} = m \cdot 2017^2$ for some positive integer m. Find the remainder when m is divided by 2017.

Problem 14. Consider the quadratic polynomial $g(x) = x^2 + x + 1020100$. A positive odd integer n is called g-friendly if and only if there exists an integer m such that n divides $2 \cdot g(m) + 2021$. Find the number of g-friendly positive odd integers less than 100.

Problem 15. Let ABC be a triangle with $AB < AC$, inscribed in a circle with radius 1 and center O. Let H be the intersection of the altitudes of ABC. Let lines $\overline{OH}, \overline{BC}$ intersect at T. Suppose there is a circle passing through B, H, O, C. Given $\cos(\angle ABC - \angle BCA) = \frac{11}{32}$, then $TO = \frac{m\sqrt{p}}{n}$ for relatively prime positive integers m, n and squarefree positive integer p. Find $m+n+p$.